Billiards in L-shaped tables with barriers

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Abstract

We compute the volumes of the eigenform loci in the moduli space of genus two Abelian differentials. From this, we obtain asymptotic formulas for counting closed billiards paths in certain L-shaped polygons with barriers.

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1 Introduction

Let $P = P(a, b, t)$ be the L-shaped polygon with barrier shown in Figure 1. A *billiards path* on P is a path which is geodesic on $\mathrm{int}(P)$ and which bounces off of the sides of P so that the angle of incidence equals the angle of reflection. We allow billiards paths to pass through one of the six corners of P but not through the endpoint p of the barrier. A *saddle connection* is a billiards path which joins p to itself. Each periodic billiards path is contained in a family of parallel periodic billiards paths called a *cylinder*, and each cylinder is bounded by a union of saddle connections. A saddle connection is *proper* if it does not

bound a cylinder. We say that a saddle connection is *multiplicity one* if it passes through a corner and *multiplicity two* otherwise.

We define the following counting functions:

$$\begin{aligned} N_{\mathrm{c}}(P,L) &= \#\{\text{maximal cylinders of length at most } L\}, \\ N_{\mathrm{s}}^1(P,L) &= \#\{\text{proper multiplicity one saddle connections of length at most } L\}, \\ N_{\mathrm{s}}^2(P,L) &= \#\{\text{proper multiplicity two saddle connections of length at most } L\}. \end{aligned}$$

Note that we are counting unoriented saddle connections and cylinders, and we do not count periodic paths which repeat more than once.

In this paper, we will show:

Theorem 1.1. *Consider the polygon* $P = P(x + z\sqrt{d}, y + z\sqrt{d}, t)$*, with* $t > 0$*,* $x, y, z \in \mathbb{Q}$*,* $x + y = 1$*,* $d \in \mathbb{N}$ *nonsquare, and*

$$P \neq P\left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{5-\sqrt{5}}{10}\right). \tag{1.1}$$

We have the following asymptotics:

$$\begin{aligned} N_{\mathrm{c}}(P,L) &\sim \frac{15}{2\pi \operatorname{Area}(P)} L^2, \\ N_{\mathrm{s}}^1(P,L) &\sim \frac{27}{16 \operatorname{Area}(P)} \pi L^2, \quad \textit{and} \\ N_{\mathrm{s}}^2(P,L) &\sim \frac{5}{16 \operatorname{Area}(P)} \pi L^2. \end{aligned}$$

Remark. The notation $a(L) \sim b(L)$ means $a(L)/b(L) \to 1$ as $L \to \infty$.

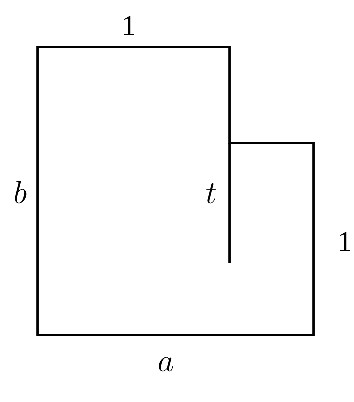

Figure 1: $P(a, b, t)$

Using [Vee89], one can show for the surface P of (1.1),

$$N_{\mathrm{c}}(P,L) \sim \frac{75}{16\pi \operatorname{Area}(P)} L^2,$$

$$N_{\mathrm{s}}^1(P,L) \sim \frac{1125+425\sqrt{5}}{128\pi\,\mathrm{Area}(P)} L^2, \quad \text{and}$$
$$N_{\mathrm{s}}^2(P,L) \sim \frac{125-25\sqrt{5}}{32\pi\,\mathrm{Area}(P)} L^2.$$

Spaces of Abelian differentials. There is an unfolding construction which associates to a rational angled polygon P a Riemann surface X equipped with an Abelian differential (holomorphic 1-form) ω. The *unfolding* (X,ω) consists of several reflected copies of P glued along their boundaries. One step in the proof of Theorem 1.1 is the computation of the volumes of certain spaces of Abelian differentials in which the unfoldings of the polygons $P(a,b,t)$ lie. These volume computations are the focus of this paper.

More precisely, let $\Omega_1\mathcal{M}_2$ be the moduli space of genus two Abelian differentials (X,ω) such that $\int_X |\omega|^2 = 1$, and let $\Omega_1\mathcal{M}_2(1,1)$ be the locus of differentials with two simple zeros. Given a square-free $D \in \mathbb{N}$ with $D \equiv 0$ or $1 \pmod 4$, let

$$\Omega_1 E_D \subset \Omega_1\mathcal{M}_2$$

be the locus of eigenforms for real multiplication of discriminant D (we discuss real multiplication in §4), and let $\Omega_1E_D(1,1) = \Omega_1E_D \cap \Omega_1\mathcal{M}_2(1,1)$. The locus Ω_1E_D is a circle bundle over a Zariski-open subset of the Hilbert modular surface of discriminant D,

$$X_D = (\mathbb{H} \times (-\mathbb{H}))/\mathrm{SL}_2\mathcal{O}_D, \tag{1.2}$$

where $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})$ is the unique real quadratic order of discriminant D. The unfolding of the polygon $P(a,b,t)$ is genus two. It has two simple zeros if and only if $t > 0$, and it is an eigenform if and only if a and b are as in Theorem 1.1.

There is a natural action of $\mathrm{SL}_2\mathbb{R}$ on $\Omega_1\mathcal{M}_2$ which preserves the spaces $\Omega_1\mathcal{M}_2(1,1)$ and Ω_1E_D. There is a natural absolutely continuous, finite, ergodic $\mathrm{SL}_2\mathbb{R}$-invariant measure μ_D^1 on Ω_1E_D which is constructed in [McM07a]. The main result of this paper is that the volume of Ω_1E_D with respect to this measure is proportional to the orbifold Euler characteristic of X_D:

Theorem 1.2. *The μ_D^1-volume of Ω_1E_D is $4\pi\chi(X_D)$.*

Remark. This theorem was conjectured by Maryam Mirzakhani.

From a formula of Siegel for $\chi(X_D)$, we obtain the explicit formula:

$$\mathrm{vol}\,\Omega_1E_{f^2D} = 8\pi f^3 \zeta_{\mathbb{Q}(\sqrt{D})}(-1) \sum_{r|f} \left(\frac{D}{r}\right) \frac{\mu(r)}{r^2},$$

for D a fundamental discriminant.

Counting functions. In order to prove Theorem 1.1, we need also to evaluate integrals over $\Omega_1 E_D(1,1)$ of certain counting functions. A Riemann surface with Abelian differential (X,ω) carries a flat metric $|\omega|$ on the complement of the zeros of ω. A *saddle connection* is a geodesic segment joining two zeros of ω whose interior is disjoint from the zeros. If X is genus two, it has a hyperelliptic involution J. We say that a saddle connection I joining distinct zeros has multiplicity one if $J(I) = I$, and otherwise it has multiplicity two. As for billiards paths, closed geodesics occur in parallel families forming metric cylinders bounded by saddle connections. We define for $T \in \Omega_1\mathcal{M}_2(1,1)$:

$$\begin{aligned} N_{\mathrm{c}}(T,L) =& \#\{\text{maximal cylinders on } T \text{ of circumference at most } L\},\\ N^1_{\mathrm{s}}(T,L) =& \#\{\text{multiplicity one saddle connections on } T \text{ of length at most } L\\ & \text{joining distinct zeros of } T\}, \quad \text{and}\\ N^2_{\mathrm{s}}(T,L) =& \#\{\text{pairs of multiplicity two saddle connections on } T \text{ of length at}\\ & \text{most } L \text{ joining distinct zeros of } T\}. \end{aligned}$$

We will show in §11:

Theorem 1.3. *For any $L > 0$,*

$$\int_{\Omega_1 E_D(1,1)} N_{\mathrm{c}}(T,L)\, d\mu^1_D(T) = 60 L^2 \chi(X_D), \tag{1.3}$$

$$\int_{\Omega_1 E_D(1,1)} N^1_{\mathrm{s}}(T,L)\, d\mu^1_D(T) = \frac{27}{2}\pi^2 L^2 \chi(X_D), \quad \textit{and} \tag{1.4}$$

$$\int_{\Omega_1 E_D(1,1)} N^2_{\mathrm{s}}(T,L)\, d\mu^1_D(T) = \frac{5}{2}\pi^2 L^2 \chi(X_D). \tag{1.5}$$

Uniform distribution of circles. Given $T \in \Omega_1 E_D$, let m_T be the measure on $\Omega_1 E_D$ obtained by transporting the Haar measure on $\mathrm{SO}_2\mathbb{R}$ to the orbit $\mathrm{SO}_2\mathbb{R}\cdot T$. Let

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

The measure $(a_t)_* m_T$ is the uniform measure on a circle centered at p in the $\mathrm{SL}_2\mathbb{R}$ orbit through p.

There is a *Teichmüller curve* $\Omega_1 D_{10} \subset \Omega_1 E_5(1,1)$ which is the $\mathrm{SL}_2\mathbb{R}$-orbit of the Abelian differential obtained by identifying opposite sides of a regular decagon. In §12, we prove

Theorem 1.4. *For any point $T \in \Omega_1 E_D(1,1)$ which does not lie on the curve $\Omega_1 D_{10}$, we have*

$$\lim_{t\to\infty} (a_t)_* m_T = \frac{\mu^1_D}{\operatorname{vol}\mu^1_D}. \tag{1.6}$$

The proof of Theorem 1.4 is a straightforward adaption of an argument in [EMM06] together with a partial classification of ergodic horocycle-invariant measures on $\Omega_1 E_D$ obtained in [CW].

It follows from [EM01] that if (1.6) holds for $T \in \Omega_1 E_D(1,1)$, then

$$N(T,L) \sim cL^2,$$

where

$$c = \frac{1}{L^2 \operatorname{vol}(\mu_D^1)} \int\limits_{\Omega_1 E_D(1,1)} N_{\mathrm{c}}(T,L)\, d\mu_D^1(T),$$

and similarly for the counting functions from saddle connections.

We then obtain from Theorems 1.2 and 1.3:

Theorem 1.5. *For every $T \in \Omega_1 E_D(1,1)$ which does not lie on $\Omega_1 D_{10}$,*

$$\begin{aligned} N_{\mathrm{c}}(T,L) &\sim \frac{15}{\pi} L^2, \\ N_{\mathrm{s}}^1(T,L) &\sim \frac{27}{8}\pi L^2, \quad \textit{and} \\ N_{\mathrm{s}}^2(T,L) &\sim \frac{5}{8}\pi L^2. \end{aligned}$$

Theorem 1.1 follows directly from this by applying the unfolding construction.

Outline of proof of Theorem 1.2. The proof of Theorem 1.2 proceeds in the following steps:

1. Let μ_D be the projection of μ_D^1 to X_D. It is enough to calculate $\operatorname{vol}(\mu_D)$. There is a foliation $\mathcal{F}_D$ of X_D by hyperbolic Riemann surfaces covered by $\mathrm{SL}_2\mathbb{R}$-orbits on $\Omega_1 E_D$. The foliation $\mathcal{F}_D$ has a natural transverse measure which we discuss in §4, and the product of this measure with the leafwise hyperbolic measure is μ_D.

2. There is a compactification Y_D of X_D which we studied in [Bai]. The space Y_D is a complex orbifold, and the boundary $\partial X_D = Y_D \setminus X_D$ is a union of rational curves. These curves are parameterized by numerical invariants called *prototypes* which we discuss in §5. There is one rational curve $C_P \subset \partial X_D$ for each prototype P. We summarize the relevant properties of Y_D in §7.

3. The foliation $\mathcal{F}_D$ doesn't extend to a foliation of Y_D; however, in §8, we show that integration over the measured foliation $\mathcal{F}_D$ defines a closed current on Y_D. The form,

$$\widetilde{\omega}_1 = \frac{1}{2\pi} \frac{dx_1 \wedge dy_1}{y_1^2},$$

on $\mathbb{H}\times\mathbb{H}$ descends to a closed 2-form ω_1 on X_D which also defines a closed current on Y_D. In §8, we also show that

$$\mathrm{vol}(\mu_D) = [\overline{\mathcal{F}}_D]\cdot[\omega_1], \tag{1.7}$$

where $[\overline{\mathcal{F}}_D]$ and $[\omega_1]$ are the classes in $H^2(Y_D;\mathbb{R})$ defined by these currents, and the product is the intersection product on cohomology.

4. There are closed leaves W_D and P_D of $\mathcal{F}_D$ whose closures are smooth curves in Y_D. From [Bai], we have

$$[\omega_1] = -\frac{1}{3}[\overline{W}_D] + \frac{3}{5}[\overline{P}_D] - \frac{4}{15}B_D,$$

where B_D is a certain linear combination of fundamental classes of the curves $C_P \subset \partial X_D$. Thus by (1.7), we need only to compute the intersection numbers of these curves with $[\overline{\mathcal{F}}_D]$.

5. In §9, we show that

$$[\overline{W}_D]\cdot[\overline{\mathcal{F}}_D] = [\overline{P}_D]\cdot[\overline{\mathcal{F}}_D] = 0. \tag{1.8}$$

Heuristically, these intersection numbers are zero because these curves are leaves of $\overline{\mathcal{F}}_D$, and a closed, nonatomic leaf of a measured foliation has zero intersection number with that foliation. Since $\overline{\mathcal{F}}_D$ is singular at the cusps of $\overline{W}_D$ and $\overline{P}_D$, this intersection number is *a priori* not necessarily zero. However, leaves close to $\overline{W}_D$ or $\overline{P}_D$ tend to diverge from these curves near the cusps, so these cusps should not contribute to the intersection number.

Here is the idea of the rigorous proof of (1.8): In §2, we construct for any Riemann surface X a canonical measured foliation $\mathcal{C}_X$ of TX by Riemann surfaces and show essentially that the intersection number of the zero section of TX with $\mathcal{C}_X$ is zero. In §9, we show that there is a tubular neighborhood of $\overline{W}_D$ or $\overline{P}_D$ which is isomorphic to a neighborhood of the zero section of (a twist of) $T\overline{W}_D$ or $T\overline{P}_D$. Furthermore, this isomorphism takes the foliation $\mathcal{F}_D$ to the canonical foliation defined in §2. From this, we obtain (1.8).

6. In §10, we calculate the intersection numbers $[\overline{\mathcal{F}}_D]\cdot C_P$ and obtain Theorem 1.2.

Notation. Here we fix some notation that we will use throughout this paper. Given a manifold M equipped with a foliation $\mathcal{F}$ by hyperbolic Riemann surfaces and a n-form ω, we will denote by $\|\omega\|_{\mathcal{F}}$ the function on M defined at x by:

$$\|\omega_x\|_{\mathcal{F}} = \sup_{v_1,\dots,v_n\in T_x\mathcal{F}} \frac{|\omega(v_1,\dots,v_n)|}{\|v_1\|\cdots\|v_n\|}, \tag{1.9}$$

where the norms of vectors is with respect the hyperbolic metric. Similarly, if h is a Riemannian metric, then $\|\omega\|_h$ will denote the analogous norm with the supremum over all vectors tangent M.

We use the notation $x \prec y$ to mean that $|x| < C|y|$ for an arbitrary positive constant C. We use $x \asymp y$ to mean that $c < |x/y| < C$ for arbitrary positive constants c and C, and we say that x is *comparable* to y. We write $C(x)$ for an arbitrary positive constant depending only on the data x.

We use the following notation for subsets of $\mathbb{C}$: $\mathbb{H}$ will denote the upper-half plane, Δ_r the disk of radius r, and $\Delta_r^* = \Delta_r \setminus \{0\}$ the punctured disk.

$\mathrm{GL}_2^+\mathbb{R}$ denotes the identity component of $\mathrm{GL}_2\mathbb{R}$. We write B for the subgroup of upper-triangular matrices, and H for the subgroup of B with ones on the diagonal. We write:

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Notes and references. McMullen classified the orbit closures for the action of $\mathrm{SL}_2\mathbb{R}$ on $\Omega_1\mathcal{M}_2$ in the series of papers, [McM06, McM05b, McM05a, McM03b, McM07a, McM03a]. See also [Cal04] and [HL06]. The orbit closures in this classification are the entire space, the stratum $\Omega_1\mathcal{M}_2(2)$, the eigenform loci $\Omega_1 E_D$, the curves $\Omega_1 W_D$, the decagon curve $\Omega_1 D_{10}$, and infinitely many Teichmüller curves generated by square-tiled surfaces. This paper, together with [Bai] almost finishes computing the volumes of all of these orbit closures (see also [LR06]). It only remains to understand the Teichmüller curves generated by square-tiled surfaces with two simple zeros.

There has been much recent work on computing asymptotics for closed geodesics and configurations of saddle connections on rational billiards tables and flat surfaces. It is conjectured for any surface (X, ω) of any genus,

$$N_{\mathrm{c}}((X,\omega), L) \sim CL^2$$

for some constant C which depends on (X, ω). This constant C is called a *Siegel-Veech constant* Veech [Vee89] showed quadratic asymptotics for any *lattice surface* (X, ω) (this means that the affine automorphism group of (X, ω) determines a lattice in $\mathrm{SL}_2\mathbb{R}$). He also computed the Siegel-Veech constants for an infinite series of examples obtained by gluing regular n-gons. Eskin and Masur [EM01], building on work of Veech [Vee98], obtained quadratic asymptotics for almost every Abelian differential (X, ω) with a given genus and orders of zeros, and [EMZ03] computed the Siegel-Veech constants. Eskin, Masur, and Schmoll [EMS03] evaluated the Siegel-Veech constants for the polygons $P(a, b, t)$ of Theorem 1.1 with $a, b \in \mathbb{Q}$ and $t \in \mathbb{R} \setminus \mathbb{Q}$ (this case can also be handled with the methods of this paper). See also [GJ00], [Sch05], [EMM06], and [Lel06] for more about Siegel-Veech constants.

I would like to thank Curt McMullen, Maryam Mirzakhani, Alex Eskin, Howard Masur, and Florian Herzig for useful conversations over the course of this work. I am also grateful to IHES for their hospitality during the preparation of this paper.

2 Canonical foliations

In this section, we discuss a canonical measured foliation $\mathcal{C}_X$ of the tangent bundle TX of any Riemann surface X. In the case where X has finite volume, we study smooth 2-forms on a certain extension L of the bundle TX over the compactification $\overline{X}$. We will show that for some compactly supported 2-form Ψ on L which represents the Thom class, $\int_{\mathcal{C}_X} \Psi = 0$. These foliated bundles will serve as models for tubular neighborhoods of closed leaves of $\mathcal{F}_D$ and will be used in the proof of (1.8) in §9.

Action of $\mathrm{PSL}_2\mathbb{R}$ **on** $\mathbb{C}$. Let Q be the complex plane $\mathbb{C}$ equipped with the quadratic differential

$$q = \frac{1}{2z} dz^2.$$

Define an action of $\mathrm{PSL}_2\mathbb{R}$ on Q by q-affine automorphisms as follows. Let $\mathrm{SL}_2\mathbb{R}$ act on $\mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x + iy) = (ax - by) + i(-cx + dy).$$

Given $z \in Q$, define $A \circ z$ so that the following diagram commutes,

$$\begin{tikzcd} \mathbb{C} \arrow[r, "A\cdot"] \arrow[d, "f"'] & \mathbb{C} \arrow[d, "f"] \\ Q \arrow[r, "A\circ"] & Q \end{tikzcd}$$

where $f(z) = z^2$.

Canonical foliations. Let $\mathrm{PSL}_2\mathbb{R}$ act on $\mathbb{H}$ on the left by Möbius transformations in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d},$$

and give $\mathbb{H} \times Q$ the product action of $\mathrm{PSL}_2\mathbb{R}$. Define $\Phi\colon \mathbb{H} \times Q \to T\mathbb{H}$ by

$$\Phi(z, (u + iv)^2) = \left(z, (u + zv)^2 \frac{\partial}{\partial z} \right).$$

The following lemma is a straightforward calculation:

Lemma 2.1. *Φ commutes with the* $\mathrm{PSL}_2\mathbb{R}$ *actions on $\mathbb{H} \times Q$ and $T\mathbb{H}$.*

The map Φ sends the horizontal foliation of $\mathbb{H} \times Q$ with the transverse measure induced by the quadratic differential q to a measured foliation of $T\mathbb{H}$. Call this measured foliation $\mathcal{C}_{\mathbb{H}}$.

Let $X = \Gamma\backslash\mathbb{H}$ be a hyperbolic Riemann surface, or more generally a hyperbolic Riemann surface orbifold. By Lemma 2.1, the foliation $\mathcal{C}_{\mathbb{H}}$ of $T\mathbb{H}$ descends to a measured foliation of TX. Call this measured foliation $\mathcal{C}_X$.

Global angular forms. Let X be a compact 2-dimensional real orbifold equipped with a line bundle $\pi\colon L \to X$ with a Hermitian metric. Given an open set $U \subset X$ with a trivialization $\pi^{-1}(U) \to U \times \mathbb{C}$, let $d\theta_U$ be the 1-form induced by the standard angular form on $\mathbb{C}$,

$$d\theta = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Choose a compactly supported smooth bump function $\rho\colon [0,\infty) \to \mathbb{R}$ with $\rho(r) \equiv 1$ near 0. Using the Hermitian metric, we regard $\rho(r)$ as a smooth function on L.

The following theorem follows from the discussion on pp.70-74 of [BT82]:

Theorem 2.2. *There is a smooth* 1*-form* ψ *on the complement of the zero section of* L *with the following properties:*

- *For any contractable domain* $U \subset X$*, we have on* $\pi^{-1}(U)$*,*

  $$\psi = d\theta_U + \pi^* \eta,$$

  for some smooth 1*-form* η *on* U*.*

- $d\psi = -\pi^* e$*, with* e *a smooth* 2*-form representing the Euler class of* L*.*

- $\Psi = d(\rho(r)\psi)$ *is a compactly supported smooth* 2*-form representing the Thom class of* L*.*

Such a form ψ is called a *global angular form* for L.

Intersection of zero section with $\mathcal{C}_X$. Now, consider the following situation. Let $X = \Gamma\backslash\mathbb{H}$ be a finite volume hyperbolic orbifold and $\overline{X}$ its compactification. Let

$$L = T\overline{X}\left(-\sum c_i\right),$$

where the c_i are the cusps of X. That is, L is the line bundle whose nonzero holomorphic sections are holomorphic sections of $T\overline{X}$ which have simple zeros at the cusps c_i. Equip L with a Hermitian metric h. Let ψ be a global angular form on L and let $\Psi = d(\rho(r)\psi)$.

Theorem 2.3. *With* X *and* Ψ *as above,*

$$\int_{\mathcal{C}_X} \Psi = 0.$$

The proof will follow a series of lemmas.

Lemma 2.4. *For any smooth, compactly supported form* η *on* L*,* $\|\eta\|_{\mathcal{C}_X} \prec 1$*.*

Proof. Since η is a smooth form on L, it suffices to bound $\|\eta\|_{\mathcal{C}_X}$ over a neighborhood of each cusp. Suppose c is a cusp of $X = \Gamma\backslash\mathbb{H}$, and conjugate Γ so that $c = i\infty$ and $\operatorname{Stab}_\infty \Gamma$ is generated by $z \mapsto z + 2\pi$.

Identify $T\mathbb{H}$ and $\mathbb{H} \times \mathbb{C}$ by

$$(z, t) \mapsto \left(z, t\frac{\partial}{\partial z}\right).$$

Identify a neighborhood of the cusp c in $\overline{X}$ and $\Delta_\epsilon \subset \mathbb{C}$ with coordinate w by $w = e^{iz}$. Identify $T\Delta_\epsilon(-0)$ with $\Delta_\epsilon \times \mathbb{C}$ by

$$(w, s) \mapsto \left(w, sw\frac{\partial}{\partial w}\right).$$

The (z, t) and (w, s)-coordinates are related by $s = -it$. We give $T\Delta_\epsilon(-0)$ the Euclidean metric e inherited from $\mathbb{C}^2$.

At $(z, t\frac{\partial}{\partial z}) \in T\mathbb{H}$, suppose $(u + zv)^2 = t$ with $u, v \in \mathbb{R}$, which implies

$$v = \frac{\operatorname{Im}\sqrt{t}}{\operatorname{Im} z}.$$

The vector,

$$\begin{aligned} V &= \operatorname{Im} z \left(\frac{\partial}{\partial z} + 2v(u + zv)\frac{\partial}{\partial t}\right) \\ &= \operatorname{Im} z\frac{\partial}{\partial z} + i(|t| - t)\frac{\partial}{\partial t}, \end{aligned} \tag{2.1}$$

is a unit tangent vector to $\mathcal{C}_X$ at $(z, t\frac{\partial}{\partial z})$. In (w, s)-coordinates on $T\Delta_\epsilon(-0)$,

$$V = -iw\log|w|\frac{\partial}{\partial w} + (|s| - is)\frac{\partial}{\partial s}$$

is a unit tangent vector to $\mathcal{C}_X$ at (w, s). We have $\|V\|_e \prec s$, so for any smooth p-form η on L, $\|\eta\|_{\mathcal{C}_X} \prec s^p$ near c. Thus $\|\eta\|_{\mathcal{C}_X} \prec 1$ when η is compactly supported. ∎

Lemma 2.5. *$\|\psi\|_{\mathcal{C}_X}$ is bounded on compact subsets of L.*

Proof. Let $\widetilde{\alpha}$ be the 1-form on $T\mathbb{H}$ defined by

$$\widetilde{\alpha} = \frac{dx}{y} + d\theta.$$

The form $\widetilde{\alpha}$ is $\mathrm{PSL}_2\mathbb{R}$-invariant, so it descends to a form α on TX. This invariance can be seen from a direct computation or more conceptually as follows. Let $T^1\mathbb{H}$ be the unit tangent bundle, and let $T^0\mathbb{H}$ be the complement in TX of the zero section with projection $p\colon T^0\mathbb{H} \to T^1\mathbb{H}$. The connection form ω (in the sense of [KN63]) of the Levi-Civita connection is an isometry-invariant form on $T^1\mathbb{H}$, and $\widetilde{\alpha} = p^*\omega$, so $\widetilde{\alpha}$ is also invariant.

In the (z,t) coordinates on $T\mathbb{H}$ from the proof of Lemma 2.4, for $r \geq 0$, let v_r be the vector $\frac{\partial}{\partial z}$ at (i,r). For $\theta \in \mathbb{R}$, the vector $e^{i\theta} v_r$ is tangent at (i,r) to $\mathcal{C}_{\mathbb{H}}$, with $\widetilde{\alpha}(e^{i\theta} v_r) \leq 1$ and equality if $\theta \equiv 0 \pmod{2\pi}$. Furthermore, every unit norm tangent vector to $\mathcal{C}_{\mathbb{H}}$ is $\mathrm{PSL}_2\mathbb{R}$-equivalent to some $e^{i\theta} v_r$. Thus we have,

$$\|\alpha\|_{\mathcal{C}_X} = 1. \tag{2.2}$$

The form $\psi - \alpha$ is smooth on TX by Theorem 2.2, so $\|\psi - \alpha\|_{\mathcal{C}_X}$ is bounded on compact sets disjoint from the fibers over the cusps, and thus $\|\psi\|_{\mathcal{C}_X}$ is bounded on such sets by (2.2).

We must now bound the norm of ψ around the cusps. As in the proof of Lemma 2.4, suppose the cusp is at $i\infty$ with $\mathrm{Stab}_{i\infty}\Gamma$ generated by $z \mapsto z + 2\pi$, and identify the bundle L restricted to a neighborhood of the cusp with $T\Delta_\epsilon(-0)$. The form $d\theta$ on $T\mathbb{H}$ is invariant under $z \mapsto z + 2\pi$, so it descends to a form β on $T\Delta_\epsilon(-0)$. From the coordinates on $T\Delta_\epsilon(-0)$ in the proof of Lemma 2.4, and the first part of Theorem 2.2, we see that $\psi - \beta$ is smooth, so has bounded norm on compact sets by Lemma 2.4. By pairing the tangent vectors v to $\mathcal{C}_{\mathbb{H}}$ in (2.1) with β, we see that $\|\beta\|_{\mathcal{C}_X} \leq 2$, thus the norm of ψ is bounded on compact subsets of $T\Delta_\epsilon(-0)$. ∎

Lemma 2.6. *For any horocycle H around a cusp c of X, the inverse image $\pi^{-1}(H) \subset L$ is transverse to $\mathcal{C}_X$, and*

$$\int_{\mathcal{H}} |\rho(r)\psi| \prec \mathrm{length}(H), \tag{2.3}$$

where the integral is over the measured foliation $\mathcal{H}$ of $\pi^{-1}(H)$ induced by the intersection with $\mathcal{C}_X$.

Proof. The transversality statement follows from the fact that every leaf of $\mathcal{C}_{\mathbb{H}}$ is the graph of a holomorphic section $\mathbb{H} \to T\mathbb{H}$.

From Lemma 2.5, we have

$$\int_{\mathcal{H}} |\rho(r)\psi| < C(\rho,\psi)\,\mathrm{vol}_{\pi^{-1}(H)}(\mathrm{support}\,\rho(r)),$$

where the volume form on $\pi^{-1}(H)$ is obtained by taking the product of the transverse measure to the foliation with the hyperbolic arclength measure on the leaves. We claim that

$$\mathrm{vol}_{\pi^{-1}(H)}\,\mathrm{support}\,\rho(r) < C(\rho)\,\mathrm{length}(H), \tag{2.4}$$

which would imply (2.3).

Let g be the hyperbolic metric on TX. The hyperbolic norm of a nonzero holomorphic section over a neighborhood of a cusp is comparable to the hyperbolic norm of the vector field $z\frac{\partial}{\partial z}$ on Δ^*,

$$\left\| z \frac{\partial}{\partial z} \right\| = \left(\log \frac{1}{|z|} \right)^{-1},$$

which is bounded. Thus g/h is bounded on X, where h is the Hermitian metric on L.

For each $x \in X$, the fiber $\pi^{-1}(x) = T_x X$ has the structure of the quadratic differential, $(\mathbb{C}, dz^2/2z)$, with the unit hyperbolic norm vectors the unit circle in $\mathbb{C}$. The intersection $\pi^{-1}(x) \cap \operatorname{support} \rho(r)$ is a disk in of radius $R(x)$ in $\mathbb{C}$ with $R(x) = Kg/h$, where

$$K = \sup\{r : \rho(r) \neq 0\}.$$

Since g/h is bounded, so is R. The volume of each disk $\pi^{-1}(x) \cap \operatorname{support} \rho(r)$ is less than $\pi R/2$, which is then bounded, and (2.4) follows. ∎

Let $T^s X$ be the set of vectors of hyperbolic length s.

Lemma 2.7. *The intersection of $T^s X$ with $\mathcal{C}_X$ is transverse, and we have*

$$\int_{\mathcal{T}(s)} |\rho(r)\psi| \prec s,$$

where $\mathcal{T}(s)$ is the induced measured foliation of $T^s X$.

Proof. At $v_s = (i, s\frac{\partial}{\partial z}) \in T^s\mathbb{H}$ for $s > 0$, the leaf of $\mathcal{C}_{\mathbb{H}}$ which is the graph of $z \mapsto (z, s\frac{\partial}{\partial z})$ is transverse to $T^s\mathbb{H}$. Then $T^s\mathbb{H}$ is transverse to $\mathcal{C}_{\mathbb{H}}$ because $\mathrm{SL}_2\mathbb{R}$ acts transitively on $T^s\mathbb{H}$.

Since $\|\rho(r)\psi\|_{\mathcal{C}_X}$ is bounded by Lemma 2.5, it is enough to show

$$\operatorname{vol} T^s X \prec s, \tag{2.5}$$

where the volume is with respect to the product of the transverse measure to $\mathcal{T}(s)$ with the hyperbolic length measure on the leaves. Let h be a segment of a leaf of $\mathcal{T}(s)$. If we identify the leaf L of $\mathcal{C}_X$ containing h with $\mathbb{H}$, then h is a segment of a horocycle in L. Let $D_h \subset L$ be the domain bounded by h and the two asymptotic geodesic rays perpendicular to h at its endpoints. A simple calculation shows $\operatorname{area}(D_h) = \operatorname{length}(h)$, so

$$\operatorname{vol} T^s X = \operatorname{vol} B_s = \frac{\pi s}{2} \operatorname{vol} X,$$

where $B_s \subset TX$ is the set of vectors of length at most s. This proves (2.5). ∎

Proof of Theorem 2.3. Let $H \subset X$ be a union of embedded horocycles bounding a neighborhood C of the cusps of X.

By Stokes' Theorem,

$$\int_{\mathcal{C}_X} \Psi = \int_{\pi^{-1}C \cup B_s} \Psi + \int_{\partial(\pi^{-1}C \cup B_s)} \rho(r)\psi.$$

We have

$$\left| \int_{\pi^{-1}C \cup B_s} \Psi \right| < \|\Psi\|^{\infty}_{\mathcal{C}_X} \operatorname{vol}((\pi^{-1}C \cap \operatorname{support} \rho(r)) \cup B_s)$$
$$< C(\rho, \psi)(\operatorname{vol} C + \operatorname{vol} B_s)$$

by Lemma 2.4. This can be made arbitrarily small by choosing C and s small.

We also have

$$\left| \int_{\partial(\pi^{-1}C \cup B_s)} \rho(r)\psi \right| \leq \int_{\pi^{-1}H} |\rho(r)\psi| + \int_{T^s X} |\rho(r)\psi|$$
$$\leq C(\rho, \psi)(\operatorname{length}(H) + s),$$

by Lemmas 2.6 and 2.7. This can also be made arbitrarily small. Thus $\int_{\mathcal{C}_X} \Psi = 0$ as claimed. ■

3 Abelian differentials

We record in this section some standard background material on Riemann surface equipped with Abelian differentials and their moduli spaces. We discuss the flat geometry defined by an Abelian differential and some surgery operations on Abelian differentials. We then discuss the action of $\mathrm{GL}_2^+\mathbb{R}$ on the moduli space $\Omega\mathcal{M}_g$ of genus g Abelian differentials.

Stable Riemann surfaces. A *stable Riemann surface* is a compact, complex, one-dimensional complex analytic space with only nodes as singularities such that each connected component of the complement of the nodes has negative Euler characteristic. One can also regard a stable Riemann surface as a finite union of finite volume Riemann surfaces together with an identification of the cusps into pairs. The *(arithmetic) genus* of a stable Riemann surface X is the genus of the nonsingular surface obtained by thickening each node to an annulus.

Stable Abelian differentials. Let X be a stable Riemann surface and X_0 the complement of the nodes. A *stable Abelian differential* on X is a holomorphic 1-form ω on X_0 such that:

- The restriction of ω to each component of X_0 has at worst simple poles at the cusps.
- At two cusps p and q which have been identified to form a node,

$$\operatorname{Res}_p \omega = -\operatorname{Res}_q \omega. \tag{3.1}$$

It follows from the Riemann-Roch Theorem that the space $\Omega(X)$ of stable Abelian differentials on X is a g-dimensional vector space, where g is the genus.

Translation structure. Given a Riemann surface X, a nonzero Abelian differential ω on X determines a metric $|\omega|$ on $X \setminus Z(\omega)$, where $Z(\omega)$ is the set of zeros of ω. If $p \in U \subset X$ with U a simply connected domain, we have a chart $\phi_{p,U} \colon U \to \mathbb{C}$ defined by

$$\phi_{p,U}(z) = \int_p^z \omega$$

satisfying $\phi^* dz = \omega$. These charts differ by translations, so the metric $|\omega|$ has trivial holonomy. In this metric, a zero of ω of order p is a cone point with cone angle $2\pi(p+1)$.

A *direction* on an Abelian differential (X, ω) is a parallel line field on $X \setminus Z(\omega)$. An Abelian differential (X, ω) has a canonical *horizontal direction*, the kernel of $\operatorname{Im}\omega$.

A surface with a flat cone metric with trivial holonomy and a choice of horizontal direction is often called a *translation surface*. We have seen that an Abelian differential (X, ω) naturally has the structure of a translation surface.

Projective Abelian differentials. A *projective Abelian differential* on a Riemann surface X is an element of the projectivization $\mathbb{P}\Omega(X)$. We will denote the projective class of a form ω by $[\omega]$. One should think of a projective Abelian differential as a translation surface without a choice of horizontal direction or scale for the metric.

Homological directions. A direction v on $(X, [\omega])$ is a *homological direction* if when we choose a representative ω of $[\omega]$ so that v is horizontal, there is a homology class $\gamma \in H_1(X; \mathbb{Z})$ so that $\omega(\gamma) \in \mathbb{R} \setminus \{0\}$.

Directed Abelian differentials. A *directed Abelian differential* is a triple $(X, [\omega], v)$, where $(X, [\omega])$ is a projective Abelian differential, and v is a direction on $(X, [\omega])$. Two such objects $(X_i, [\omega_i], v_i)$ are equivalent if there is an isomorphism $f \colon X_1 \to X_2$ such that $f^*[\omega_2] = [\omega_1]$ and $f^* v_2 = v_1$. We can regard an Abelian differential (X, ω) as directed by taking the horizontal direction.

Cylinders. Every closed geodesic on an Abelian differential (X, ω) is contained in a family of parallel closed geodesics isometric to a flat metric cylinder $S^1 \times (a, b)$. The maximal such cylinder is bounded by a union of saddle connections.

A (directed) Abelian differential (X, ω) is *periodic* if every leaf of the horizontal foliation which does not meet a zero of ω is closed. In this case, every leaf which meets a zero is a saddle connection. The union of all such leaves is called the *spine* of (X, ω). The complement of the spine is a union of maximal cylinders. We call (X, ω) a n-cylinder surface if the complement of the spine has n cylinders.

If (X, ω) is a stable Abelian differential, and the residue of ω at a node n is real and nonzero, then (X, ω) has two horizontal, half-infinite cylinders facing

the node n. When we speak of an n-cylinder surface, we count both of these half-infinite cylinders as one cylinder.

Connected sum. Consider two Abelian differentials (X_i, ω_i). Let $I \subset \mathbb{C}$ be a geodesic segment with two embeddings $\iota_i \colon I \to X_i \setminus Z(\omega_i)$ which are local translations. The *connected sum* of (X_1, ω_1) and (X_2, ω_2) along I is the Abelian differential obtained by cutting each (X_i, ω_i) along ι_i and regluing the two slits by local translations to obtain a connected Abelian differential.

Now suppose (X, ω) is a genus two stable Abelian differential with one separating node. The differential (X, ω) is the union of two genus one differentials (E_1, ω_1) and (E_2, ω_2) joined at a single point. Let $I \subset \mathbb{C}$ be a segment which embeds in each (E_i, ω_i) by local translations. We write

$$\mathrm{Sum}((X, \omega), I),$$

for the connected sum of the (E_i, ω_i) along I. Note that the choice of embedding $\iota_i \colon I \to E_i$ is irrelevant because the automorphism group of (E_i, ω_i) is transitive. This connected sum is a genus two Abelian differential with two simple zeros.

Splitting a double zero. Consider an Abelian differential (X, ω) with a double zero p. There is a surgery operation which we call *splitting* which replaces (X, ω) with a new differential where p becomes two simple zeros. Consider a geodesic segment $I = \overline{0w} \subset \mathbb{C}$. An *embedded "X"* is a collection of four embeddings $\iota_1, \ldots, \iota_4$ of I on (X, ω) with the following properties:

- Each ι_i is a local translations.
- Each ι_i sends 0 and no other point of I to p.
- The segments $\iota_i(I)$ are disjoint except at p.
- The segments $I_i = \iota_i(I)$ meet at p with angles

$$\angle I_1 I_2 = \pi, \quad \angle I_2 I_3 = 2\pi, \quad \text{and} \quad \angle I_3 I_4 = \pi.$$

Given E an embedded "X", we define the splitting

$$\mathrm{Split}((X, \omega), E)$$

to be the form obtained by cutting along E and regluing as indicated in Figure 2.

Collapsing a saddle connection. The operations of forming a connected sum and splitting a double zero have inverses which we call collapsing a saddle connection. Let (X, ω) be a genus two differential with two simple zeros and with $J \colon X \to X$ its hyperelliptic involution. Given a saddle connection I joining

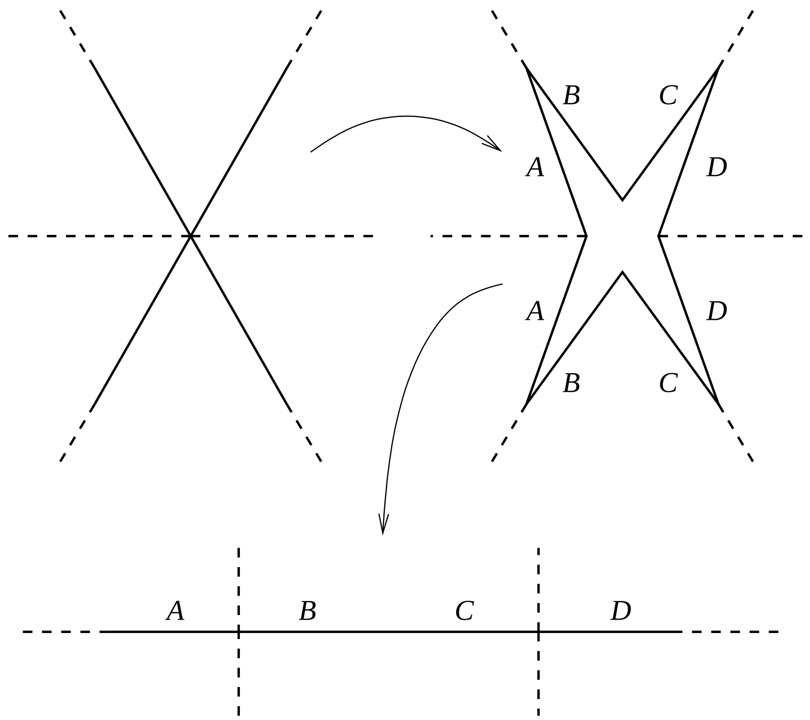

Figure 2: Splitting a double zero

distinct zeros such that $J(I) \neq I$, the union $J(I) \cup I$ is homologous to zero and so separates X into two genus one components. Let

$$\mathrm{Collapse}((X, \omega), I)$$

be the stable differential with one separating node obtained by cutting (X, ω) along I, regluing the boundary components to obtain two genus one forms, and then identifying these two genus one forms at a single point.

Now suppose I is a saddle connection of length ℓ joining zeros p_1 and p_2 such that $J(I) = I$. We say that I is *unobstructed* if there are embedded geodesic segments $I_1, I_2 \subset X$ of length $\ell/2$ with the following properties:

- I_j has one endpoint on p_j and is otherwise disjoint from the zeros and I
- I_j and I meet at an angle of 2π at p_j.
- I_1 and I_2 are disjoint.

If I is unobstructed, then we can cut along $I_1 \cup I_2 \cup I$ and then reglue following Figure 2 in reverse to obtain a new Abelian differential with two simple zeros. We also denote this differential by $\mathrm{Collapse}((X, \omega), I)$.

Note that a saddle connection is always unobstructed if it is the unique shortest saddle connection up to the action of J, so any such saddle connection can be collapsed.

Plumbing a cylinder. Let (X, ω) be a stable Abelian differential with a node n where ω has a simple pole. There are two half-infinite cylinders C_i facing n; let γ_i be a closed geodesic in C_i. The condition (3.1) ensures that the γ_i have the same holonomy with respect to the translation structure. Thus we can cut (X, ω) along the γ_i and reglue to obtain a new differential (X', ω') where the node n has been replaced with a finite cylinder. We call this operation *plumbing a cylinder*. This operation depends on two real parameters: the height of the resulting cylinder and a twist parameter which determines the gluing of the γ_i.

Teichmüller space. Let $\mathcal{T}_g$ be the Teichmüller space of closed genus g Riemann surfaces, and let $\Omega\mathcal{T}_g$ be the trivial complex vector bundle whose fiber over a Riemann surface X is the space of nonzero Abelian differentials on X.

There is a natural stratification of $\Omega\mathcal{T}_g$. Given a partition $\mathbf{n} = (n_1, \ldots, n_r)$ of $2g - 2$, let

$$\Omega\mathcal{T}_g(\mathbf{n}) \subset \Omega\mathcal{T}_g$$

be the locus of forms having zeros of orders given by the n_i. By [Vee90], the strata $\Omega\mathcal{T}_g(\mathbf{n})$ are complex submanifolds of $\Omega\mathcal{T}_g$.

Period coordinates. There are simple holomorphic coordinates on the strata $\Omega\mathcal{T}_g(\mathbf{n})$ defined in terms of periods. A form $(X, \omega) \in \Omega\mathcal{T}_g(\mathbf{n})$ defines a cohomology class in $H^1(X, Z(\omega); \mathbb{C})$. Over a contractable neighborhood $U \subset \Omega\mathcal{T}_g(\mathbf{n})$ of (X, ω), the bundle of surfaces with marked points whose fiber over (Y, η) is the surface Y marked by the points of $Z(\eta)$ marked is topologically trivial. Thus we can canonically identify the groups $H^1(Y, Z(\eta); \mathbb{C})$ as (Y, η) varies over U (this is called the Gauss-Manin connection). This defines a map

$$\phi\colon U \to H^1(X, Z(\omega); \mathbb{C}) \cong \mathbb{C}^n.$$

Theorem 3.1 ([Vee90])**.** *The maps ϕ_U are biholomorphic coordinate charts.*

Moduli space. Let $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$ be the moduli space of genus g Riemann surfaces, where Γ_g is the mapping class group. The moduli space of genus g Abelian differentials is the quotient

$$\Omega\mathcal{M}_g = \Omega\mathcal{T}_g/\Gamma_g.$$

It is a holomorphic rank g orbifold vector bundle over $\mathcal{M}_g$ and is stratified by the suborbifolds,

$$\Omega\mathcal{M}_g(\mathbf{n}) = \Omega\mathcal{T}_g(\mathbf{n})/\Gamma_g.$$

Let $\Omega_1\mathcal{M}_g$ be the locus of forms (X, ω) of unit norm $\|\omega\| = 1$, where

$$\|\omega\| = \int_X |\omega|^2.$$

Similarly, if ΩX denotes any space of Abelian differentials, $\Omega_1 X$ will denote the unit area differentials.

Deligne-Mumford compactification. Let $\overline{\mathcal{M}}_g$ denote the moduli space of genus g stable Riemann surfaces, the Deligne-Mumford compactification of $\mathcal{M}_g$. The bundle $\Omega\mathcal{M}_g$ over $\mathcal{M}_g$ extends to a bundle $\Omega\overline{\mathcal{M}}_g$ over $\overline{\mathcal{M}}_g$, the moduli space of stable Abelian differentials.

Action of $\mathrm{GL}_2^+\mathbb{R}$. There is a natural action of $\mathrm{GL}_2^+\mathbb{R}$, on $\Omega\mathcal{T}_g$. Let (X,ω) be an Abelian differential with an atlas of charts $\{\phi_\alpha\colon U \to \mathbb{C}\}$ covering $X \setminus Z$ such that $\phi_\alpha^* dz = \omega$. Given $A \in \mathrm{GL}_2^+\mathbb{R}$, define a new form $A \cdot (X,\omega)$ by pulling back the conformal structure on $\mathbb{C}$ and form dz via the new atlas $\{A \circ \phi_\alpha\}$, where A acts on $\mathbb{C} \cong \mathbb{R}^2$ in the usual way.

This defines a free action of $\mathrm{GL}_2^+\mathbb{R}$ on $\Omega\mathcal{T}_g$. This action commutes with the action of Γ_g and preserves the stratification, so we obtain an action of $\mathrm{GL}_2^+\mathbb{R}$ on $\Omega\mathcal{M}_g(\mathbf{n})$ and its strata.

The action of $\mathrm{SL}_2\mathbb{R}$ on these spaces preserve the loci of unit area forms.

The $\mathrm{GL}_2^+\mathbb{R}$ action naturally extends to an action on $\Omega\overline{\mathcal{M}}_g$ which is no longer free.

Projectivization. It will be convenient to work with the projectivized moduli spaces $\mathbb{P}\Omega\mathcal{M}_g$ of projective Abelian differentials. The foliation of $\Omega_1\mathcal{M}_g$ by $\mathrm{SL}_2\mathbb{R}$-orbits descends to a foliation $\mathcal{F}$ of $\mathbb{P}\Omega\mathcal{M}_2$ by hyperbolic Riemann surfaces.

4 Real multiplication

In this section, we discuss background material on real multiplication, eigenforms for real multiplication, and Hilbert modular surfaces. We also introduce the eigenform loci in $\Omega\mathcal{M}_2$ and the Teichmüller curves that they contain.

Orders. A *real quadratic discriminant* is a positive integer D with $D \equiv 0$ or $1 \pmod 4$. In this paper, we will also require our quadratic discriminants to by nonsquare. A quadratic discriminant D is *fundamental* if D is not of the form f^2E with E a quadratic discriminant and $f > 1$ an integer.

Given a quadratic discriminant D, let $\mathcal{O}_D$ be the ring,

$$\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c),$$

where $b, c \in \mathbb{Z}$ with $b^2 - 4c = D$. The ring $\mathcal{O}_D$ depends only on D and is the unique *quadratic order* of discriminant D.

Let $K_D \cong \mathbb{Q}(\sqrt{D})$ be the quotient field of $\mathcal{O}_D$. If D is fundamental, then $\mathcal{O}_D$ is the ring of integers in K_D.

There is an inclusion $\mathcal{O}_D \to \mathcal{O}_E$ if and only if $D = f^2E$ for some integer f and quadratic discriminant E.

We fix for the rest of this paper two embeddings $\iota_i \colon K_D \to \mathbb{R}$ with $i = 1, 2$. We will sometimes implicitly assume that $K_D \subset \mathbb{R}$ via the first embedding ι_1. We will also use the notation $\lambda^{(i)} = \iota_i(\lambda)$ for $\lambda \in K_D$.

Real multiplication. Consider a polarized Abelian surface $A = \mathbb{C}^2/\Lambda$. *Real multiplication* by $\mathcal{O}_D$ on A is a embedding of rings $\rho \colon \mathcal{O}_D \to \operatorname{End} A$ with the following properties:

- ρ is self-adjoint with respect to the polarization (a symplectic form on $\mathbb{C}^2$).
- ρ does not extend to some $\mathcal{O}_E \supset \mathcal{O}_D$.

Eigenforms. Given a form $(X, \omega) \in \Omega\mathcal{M}_2$, we say that (X, ω) is an *eigenform for real multiplication* by $\mathcal{O}_D$ if $\operatorname{Jac}(X)$ admits real multiplication

$$\rho \colon \mathcal{O}_D \to \operatorname{End} \operatorname{Jac}(X)$$

with ω as an eigenform. More precisely, the self-adjointness of ρ ensures an eigenspace decomposition of $\Omega(X)$,

$$\Omega(X) = \Omega \operatorname{Jac}(X) = \Omega^1(X) \oplus \Omega^2(X), \tag{4.1}$$

and ω is an eigenform if it lies in one of the $\Omega^i(X)$.

Given a choice of real multiplication ρ on $\operatorname{Jac}(X)$, the eigenspaces $\Omega^i(X)$ correspond naturally to embeddings of $\mathcal{O}_D$. We say that a form ω_i is an i-eigenform if

$$\omega(\rho(\gamma) \cdot \lambda) = \lambda^{(i)} \omega(\lambda)$$

for every $\gamma \in H_1(X; \mathbb{Z})$ and $\lambda \in \mathcal{O}_D$. We let $\Omega^i(X)$ be the space of i-eigenforms.

Eigenform locus. Given a real quadratic discriminant D, let

$$\Omega E_D \subset \Omega\mathcal{M}_2$$

be the locus of eigenforms for real multiplication by $\mathcal{O}_D$. Let

$$\Omega W_D = \Omega E_D \cap \Omega\mathcal{M}_2(2),$$

the locus of eigenforms for real multiplication with a double zero. McMullen proved in [McM03a]:

Theorem 4.1. *The spaces ΩE_D and ΩW_D are closed,* $\mathrm{GL}_2^+\mathbb{R}$*-invariant suborbifolds of $\Omega\mathcal{M}_2$.*

Let E_D and $W_D \subset \mathbb{P}\Omega\mathcal{M}_2$ be the respective projectivizations. By [McM03a], W_D is a (possibly disconnected) curve which is a union of closed leaves of the foliation $\mathcal{F}$ of $\mathbb{P}\Omega\mathcal{M}_2$. Such closed leaves in $\mathbb{P}\Omega\mathcal{M}_g$ are called *Teichmüller curves* because they project to curves in $\mathcal{M}_g$ which are isometrically embedded with respect to the Teichmüller metric on $\mathcal{M}_g$.

Let $\Omega E_D(1,1) = \Omega E_D \setminus \Omega W_D$, the locus of eigenforms with two simple zeros, and let $E_D(1,1)$ be its projectivization.

Hilbert modular surfaces. Let $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) \subset \mathrm{SL}_2 K_D$ be the subgroup preserving the lattice $\mathcal{O}_D \oplus \mathcal{O}_D^\vee$, where $\mathcal{O}_D^\vee$ is the *inverse different*,

$$\mathcal{O}_D^\vee = \frac{1}{\sqrt{D}}\mathcal{O}_D.$$

Concretely,

$$\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 K_D : a, d \in \mathcal{O}_D, b \in (\mathcal{O}_D^\vee)^{-1}, c \in \mathcal{O}_D^\vee \right\}.$$

$\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ is the group of symplectic $\mathcal{O}_D$-linear automorphisms of $\mathcal{O}_D \oplus \mathcal{O}_D^\vee$, where $\mathcal{O}_D \oplus \mathcal{O}_D^\vee$ is equipped with the symplectic form,

$$\langle u, v \rangle = \mathrm{Tr}_{\mathbb{Q}}^{K_D}(u \wedge v)$$

with

$$(u_1, u_2) \wedge (v_1, v_2) = u_1 v_2 - u_2 v_1.$$

The *Hilbert modular surface* X_D is the quotient,

$$X_D = \mathbb{H} \times \mathbb{H} / \mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee),$$

where $\mathrm{SL}_2 K_D$ acts on the ith factor of $\mathbb{H} \times \mathbb{H}$ by Möbius transformations using the ith embedding, $\mathrm{SL}_2 K_D \to \mathrm{SL}_2 \mathbb{R}$. This definition of X_D is equivalent to the one in (1.2).

Theorem 4.2. *X_D is the moduli space for principally polarized Abelian surfaces with a choice of real multiplication by $\mathcal{O}_D$.*

Sketch of proof. Let $\widetilde{X}_D$ be the space of triples (A, ρ, ϕ), where A is a principally polarized Abelian surface, $\rho\colon \mathcal{O}_D \to \operatorname{End} A$ is real multiplication of $\mathcal{O}_D$ on A, and $\phi\colon \mathcal{O}_D \oplus \mathcal{O}_D^\vee \to H_1(A; \mathbb{Z})$ is a symplectic isomorphism of $\mathcal{O}_D$-modules.

Let $\{\alpha, \beta\}$ be a basis of $\mathcal{O}_D \oplus \mathcal{O}_D^\vee$ over $\mathcal{O}_D$ with $\langle \alpha, \beta \rangle = 1$. Define a map $\Phi\colon \widetilde{X}_D \to \mathbb{H} \times \mathbb{H}$, which can be shown to be an isomorphism, by

$$\Phi(A, \rho, \phi) = \left(\frac{\omega_1(\beta)}{\omega_1(\alpha)}, \frac{\omega_2(\beta)}{\omega_2(\alpha)} \right),$$

where $\omega_i \in \Omega A$ is a nonzero i-eigenform.

Forgetting the marking ϕ, the map Φ descends to an isomorphism between the space of pairs (A, ρ) and X_D. ∎

See [BL04] or [McM07b] for a detailed proof of this theorem.

Partial compactification. Define the partial Deligne-Mumford compactification $\widetilde{\mathcal{M}}_2$ of $\mathcal{M}_2$ to be the open subvariety of $\overline{\mathcal{M}}_2$ obtained by adjoining to $\mathcal{M}_2$ those points of $\overline{\mathcal{M}}_2$ representing two elliptic curves joined at a node.

Let $\mathcal{A}_2$ be the Siegel modular variety parameterizing principally polarized Abelian surfaces. There is a natural morphism $\mathrm{Jac}\colon \widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ associating a surface X to its Jacobian $\mathrm{Jac}(X)$, where the Jacobian of two elliptic curves joined at a node is defined to be the product of those elliptic curves. It is a well-known fact that Jac is an isomorphism. We sketched a proof of this in [Bai, Proposition 5.4].

Embedding of X_D. There are natural embeddings $j_i\colon X_D \to \mathbb{P}\Omega\widetilde{\mathcal{M}}_2$ defined by

$$j_i(A, \rho) = (X, [\omega_i]),$$

where X is the unique Riemann surface with $\mathrm{Jac}(X) \cong A$, and ω_i is an i-eigenform for ρ. In this paper, we will regard X_D to be embedded in $\mathbb{P}\Omega\widetilde{\mathcal{M}}_2$ by j_1. So one can think of a point in X_D as representing a either an Abelian variety with a choice of real multiplication or a projective Abelian differential $(X, [\omega])$ which is an eigenform for real multiplication by $\mathcal{O}_D$.

For any eigenform $(X, [\omega]) \in X_D$, there are two choices of real multiplication $\rho\colon \mathcal{O}_D \to \mathrm{Jac}(X)$ realizing ω as an eigenform. These choices are related by the Galois involution of $\mathcal{O}_D$. We always choose ρ so that ω is a 1-eigenform

We identify E_D with $j_1^{-1}(\mathbb{P}\Omega\mathcal{M}_2) \subset X_D$. The bundle ΩE_D extends to a line bundle ΩX_D over X_D whose fiber over $(X, [\omega])$ is $\Omega^1(X)$, the forms in the projective class $[\omega]$. One can think of ΩX_D as the bundle of choices of scale for the metric on $(X, [\omega])$.

Product locus. Define the *product locus* $P_D \subset X_D$ to be the locus of points in X_D which represent two elliptic curves joined to a node. Alternatively, in terms of Abelian surfaces, P_D is the locus of Abelian surfaces in X_D which are polarized products of elliptic curves.

In the universal cover $\mathbb{H} \times \mathbb{H}$ of X_D, the curve P_D is a countable union of graphs of Möbius transformations.

As a subset of X_D, we have $E_D = X_D \setminus P_D$. Thus it is a Zariski-open subset of X_D. As $W_D \subset E_D$, the curves W_D and P_D are disjoint.

For more information about the curve P_D, see [McM07b] (where it is called $X_D(1)$) and [Bai].

Involution of X_D. The involution $\widetilde{\tau}(z_1, z_2) = (z_2, z_1)$ of $\mathbb{H} \times \mathbb{H}$ descends to an involution τ of X_D. On the level of Abelian varieties with real multiplication, τ sends the pair (A, ρ) to (A, ρ'), where ρ' is the Galois conjugate real multiplication.

On the level of projective Abelian differentials, we have $\tau(X, [\omega]) = (X, [\omega'])$, where ω' is an eigenform in the eigenspace which does not contain ω. Since τ does not change the underlying stable Riemann surface, we obtain

Proposition 4.3. *The involution τ satisfies $\tau(P_D) = P_D$.*

The $\mathrm{SL}_2\mathbb{R}$-orbit foliation. We have the foliation $\mathcal{F}$ of $\mathbb{P}\Omega\widetilde{\mathcal{M}}_2$ whose leaves are the images of $\mathrm{GL}_2^+\mathbb{R}$ orbits in $\Omega\widetilde{\mathcal{M}}_2$. By Theorem 4.1, X_D is saturated with respect to this foliation, meaning that a leaf which intersects X_D is contained in X_D. Thus there is an induced foliation $\mathcal{F}_D$ of X_D by Riemann surfaces.

The foliation $\mathcal{F}_D$ was defined in [McM07b] where it was shown to be transversely quasiconformal and given a transverse invariant measure, which we will define below.

Kernel foliation. The vertical foliation of $\mathbb{H} \times \mathbb{H}$ is preserved by the action of $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and so descends to a foliation of X_D. We will call this foliation $\mathcal{A}_D$.

Let $\Omega_1\mathcal{A}_D$ be the foliation of $\Omega_1 X_D$ obtained by pulling back $\mathcal{A}_D$ by the natural projection. The foliations $\Omega_1\mathcal{A}_D$ and $\mathcal{A}_D$ are characterized by the following property, proved in [Bai, Proposition 2.10] and [McM07b].

Proposition 4.4. *The absolute periods of forms in $\Omega_1 X_D$ are constant along the leaves of $\Omega_1\mathcal{A}_D$. The absolute periods of 1-eigenforms are constant along the leaves of $\mathcal{A}_D$ up to the action of $\mathbb{C}^*$.*

The foliation $\mathcal{A}_D$ is transverse to $\mathcal{F}_D$ (see [McM07b, Theorem 8.1]).

There is a foliation of $\Omega_1\mathcal{M}_g$ along which absolute periods of forms are constant, called the kernel foliation. Proposition 4.4 implies that $\Omega_1\mathcal{A}_D$ is the foliation of $\Omega_1 X_D$ induced by the kernel foliation.

Transverse measure. We now define a canonical quadratic differential along the leaves of $\Omega_1\mathcal{A}_D$ and use it to define a transverse invariant measure to $\mathcal{F}_D$ following [McM07b].

Let $f\colon \Omega_1 E_D(1,1) \to \mathbb{C}$ be the multivalued holomorphic function defined by

$$f(X,\omega) = \int_p^q \omega,$$

where p and q are the zeros of ω. The function f is multivalued because there is a choice of ordering of the zeros and a choice of path between them. Define

$$q = (\partial f)^2,$$

where the derivative is along leaves of $\Omega_1\mathcal{A}_D$. Since the absolute periods are constant along $\Omega_1\mathcal{A}_D$, the form q is well defined. The form q is a meromorphic quadratic differential on $\Omega_1 X_D$ along the leaves of $\Omega_1\mathcal{A}_D$ which is zero along $\Omega_1 W_D$, has simple poles along $\Omega_1 P_D$, and is elsewhere nonzero and finite (see [McM07b] or [Bai, §10] for proofs).

The measure $|q|$ on leaves of $\Omega_1\mathcal{A}_D$ is invariant under the action of $\mathrm{SO}_2\mathbb{R}$, so it descends to a leafwise measure $|q|$ on $\mathcal{A}_D$.

Theorem 4.5 ([McM07b])**.** *The leafwise measure $|q|$ defines a transverse, holonomy invariant measure to $\mathcal{F}_D$.*

Let μ_D be the measure on X_D (supported on $E_D(1,1)$) defined by taking the product of the transverse measure $|q|$ with the leafwise hyperbolic metric on $\mathcal{F}_D$.

We say that a measure μ on $\mathbb{P}\Omega\mathcal{M}_g$ is invariant if it is the push-forward of a $\mathrm{SL}_2\mathbb{R}$-invariant measure on $\Omega_1\mathcal{M}_g$, or equivalently if it is the product of a transverse invariant measure with the leafwise hyperbolic metric. In this sense, μ_D is invariant. McMullen proved in [McM07a]:

Theorem 4.6. *The measure μ_D is finite and ergodic. Moreover, it is the only such invariant measure supported on $E_D(1,1)$.*

Remark. The invariant measure in [McM07a] was actually defined in a slightly different way, but it is not hard to see directly that the two are the same up to a constant multiple independent of D.

We also give $\Omega_1 E_D(1,1)$ the unique $\mathrm{SL}_2\mathbb{R}$-invariant measure μ_D^1 such that $\pi_*\mu_D^1 = \mu_D$. This is the product of μ_D with the uniform measure of unit mass on the circle fibers of π.

Euler characteristic of X_D. Siegel calculated the orbifold Euler characteristic $\chi(X_D)$ in terms of values of the Dedekind zeta function ζ_{K_D} of K_D. He showed in [Sie36] (see also [Bai, Theorem 2.12]):

Theorem 4.7. *If D is a fundamental discriminant, and $f \in \mathbb{N}$, then*

$$\chi(X_{f^2D}) = 2f^3\zeta_{K_D}(-1)\sum_{r|f}\left(\frac{D}{r}\right)\frac{\mu(r)}{r^2}. \tag{4.2}$$

Remark. Here, $\left(\frac{D}{r}\right)$ is the Kronecker symbol, defined in [Miy89], and μ is the Möbius function.

For a fundamental discriminant D, Cohen defined in [Coh75]:

$$H(2,f^2D) = -12\zeta_{K_D}(-1)\sum_{r|f}\mu(r)\left(\frac{D}{r}\right)r\sigma_3\left(\frac{f}{r}\right), \tag{4.3}$$

where

$$\sigma_m(n) = \sum_{d|n} d^m.$$

From Möbius inversion, and (4.2), we obtain:

$$\sum_{r|f}\chi(X_{r^2D}) = -\frac{1}{6}H(2,f^2D).$$

Cohen proved in [Coh75] (see also [Sie69]):

Theorem 4.8. *For any real quadratic discriminant D,*

$$H(2,D) = -\frac{1}{5}\sum_{e\equiv D\ (2)}\sigma_1\left(\frac{D-e^2}{4}\right).$$

Cusps of X_D. Given $\sigma \in \mathbb{P}^1(K_D)$ and $r > 0$, define

$$N_r(\sigma) = A^{-1}\{x_j + iy_j \in \mathbb{H}\times\mathbb{H} : y_1y_2 > r\}$$

for some $A \in \mathrm{SL}_2 K_D$ such that $A\sigma = \infty$. The set $N_r(\sigma)$ is independent of the choice of A. Let Γ_σ be the stabilizer of σ in $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$, and define

$$\mathfrak{N}_r(\sigma) = N_r(\sigma)/\Gamma_\sigma.$$

We call a $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$-orbit in K_D a *cusp* of X_D. Let $C(X_D) \subset \mathbb{P}^1(X_D)$ be a set of representatives for the set of cusps of X_D.

Theorem 4.9. *If r is sufficiently large, then for each $\sigma \in \mathbb{P}^1(K_D)$, the natural embedding $\mathfrak{N}_r(\sigma) \to X_D$, is injective. Furthermore, X_D is the disjoint union,*

$$X_D = K \cup \bigcup_{\sigma \in C(X_D)} \mathfrak{N}_r(\sigma),$$

where K is a compact submanifold with boundary which is a deformation retract of X_D.

Proof. See the discussion on pp. 7-11 of [vdG88]. ■

X_D can be compactified by adding one point for each cusp of X_D. The sets $\mathfrak{N}_r(\sigma)$ form a neighborhood basis of the cusp σ.

Given a cusp σ of X_D, define the covering,

$$X_D^\sigma = \mathbb{H} \times \mathbb{H}/\Gamma_\sigma.$$

Full $\mathcal{O}_D$-modules. Given an Abelian surface with real multiplication $(A, \rho) \in X_D$, we say that an $\mathcal{O}_D$-submodule $M \subset H_1(A; \mathbb{Z})$ is *full* if for any $x \in M$ and $n \in \mathbb{Z}$, we have $nx \in M$ if and only if $x \in M$. We say that the rank of M is the dimension of $M \otimes \mathbb{Q}$ as a K_D-vector space. Let $\mathcal{X}_D$ be the disjoint union,

$$\mathcal{X}_D = \bigcup_{c \in C(X_D)} X_D^c.$$

Proposition 4.10. *$\mathcal{X}_D$ is the moduli space of all triples (A, ρ, M) with $(A, \rho) \in X_D$ and $M \subset H_1(A; \mathbb{Z})$ a full, rank one $\mathcal{O}_D$-submodule.*

Proof. Let $[x : y] \in \mathbb{P}^1(K_D)$. By the proof of Theorem 4.2, $X_D^{[x:y]}$ is the moduli space of triples (A, ρ, ϕ), where A is a principally polarized Abelian surface, ρ is real multiplication by $\mathcal{O}_D$, and ϕ is a symplectic isomorphism $\phi\colon \mathcal{O}_D \oplus \mathcal{O}_D^\vee \to H_1(A; \mathbb{Z})$ defined up to the action of $\Gamma_{[x:y]}$ on $\mathcal{O}_D \oplus \mathcal{O}_D^\vee$.

Given such a triple $(A, \rho, \phi) \in X_D^{[x:y]}$, define $M \subset H_1(A; \mathbb{Z})$ as follows. Define

$$M_{[x:y]} = K_D \cdot (x, y) \cap \mathcal{O}_D \oplus \mathcal{O}_D^\vee.$$

Since $M_{[x:y]}$ is preserved by the action of $\Gamma_{[x:y]}$, we can define $M = \phi(M_{[x:y]})$.

Conversely, given a triple (A, ρ, M), choose a symplectic $\mathcal{O}_D$-isomorphism $\psi\colon \mathcal{O}_D \oplus \mathcal{O}_D^\vee \to H_1(A; \mathbb{Z})$. Then $\psi^{-1}(M) = M_{[x:y]}$ for some $[x : y] \in \mathbb{P}^1(K_D)$. Composing ψ with an element of $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$, we can assume $[x : y] \in C(X_D)$. The marking ψ is then unique up to composition with elements of $\Gamma_{[x:y]}$. Then (A, ρ, ψ) yields a well-defined point in $X_D^{[x:y]}$. ■

Homological directions. $\mathcal{X}_D$ also parameterizes eigenforms for real multiplication equipped with a homological direction:

Proposition 4.11. *$\mathcal{X}_D$ is the moduli space of directed Abelian differentials $(X, [\omega], v)$ with $(X, [\omega]) \in X_D$ an eigenform for real multiplication and v a homological direction.*

Proof. Let $(X, [\omega]) \in X_D$. Using Proposition 4.10, we need only to show that there is a natural correspondence between full, rank one $\mathcal{O}_D$-modules $M \subset H_1(X; \mathbb{Z})$ and homological directions on $(X, [\omega])$.

Given such an M, choose a representative ω of $[\omega]$ so that $\omega(\gamma) \in \mathbb{R}$ for some $\gamma \in M$. Since M is rank one, for any $\gamma_1 \in M$, we have $\gamma_1 = \lambda \cdot \gamma$ for some $\lambda \in K_D$, so $\omega(\gamma_1) = \lambda^{(1)}\omega(\gamma)$. Thus $\omega(\gamma_1) \in \mathbb{R}$ for any $\gamma_1 \in M$, and so this normalization of $[\omega]$ is canonically determined by M. We then set v to be the horizontal direction of ω.

Conversely, if v is a homological direction on $(X, [\omega])$, choose a representative ω of $[\omega]$ so that v is horizontal, and let

$$M = \{\gamma \in H_1(X; \mathbb{Z}) : \omega(\gamma) \in \mathbb{R}\}.$$

M is clearly a full $\mathcal{O}_D$-submodule, and $M \neq H_1(X; \mathbb{Z})$ because the periods can not all be real by the Riemann relations. Thus M is rank one. ■

Detecting eigenforms. We conclude this section by showing that eigenforms for real multiplication in genus two can be detected in terms of their periods.

Proposition 4.12. *Let A be a principally polarized Abelian surface with a holomorphic 1-form $\omega \in \Omega(A)$. Suppose there is a monomorphism $\rho \colon \mathcal{O}_D \to \operatorname{End} H_1(A; \mathbb{Z})$ with the following properties:*

- *ρ is self-adjoint with respect to the polarization.*
- *ρ doesn't extend to a larger order $\mathcal{O}_E \supset \mathcal{O}_D$.*
- *$\omega(\rho(\lambda) \cdot x) = \lambda^{(i)}\omega(x)$ for every $x \in H_1(A; \mathbb{Z})$ and $\lambda \in \mathcal{O}_D$.*

Then ρ defines real multiplication by $\mathcal{O}_D$ on A with ω an i-eigenform.

Proof. An endomorphism $\rho(\lambda)$ determines a real-linear endomorphism T of $H_1(A; \mathbb{R}) \cong \Omega(A)^*$. The dual endomorphism T^* of $\Omega(A)$ is self-adjoint, preserves the complex line spanned by ω, and is complex linear on this line. Thus T^* is complex linear by a linear algebra argument (see [McM03a, Lemma 7.4]), and so T is complex linear as well. Therefore T defines real multiplication on $\mathcal{O}_D$, with ω an eigenform. ■

5 Prototypes

We now introduce a combinatorial object called a *prototype*, which is closely related to the combinatorics of various subsets of Hilbert modular surfaces and

their compactifications. Our prototypes are nearly the same as the splitting prototypes introduced in [McM05a]. What we call a prototype here was called a Y_D-prototype in [Bai].

Definition. A *prototype* of discriminant D is a quadruple, $(a, b, c, \bar{q})$ with $a, b, c \in \mathbb{Z}$ and $\bar{q} \in \mathbb{Z}/\gcd(a, b, c)$ which satisfies the following five properties:

$$b^2 - 4ac = D, \quad a > 0, \quad c < 0, \quad \gcd(a, b, c, \bar{q}) = 1, \quad \text{and} \quad a + b + c < 0.$$

We let $\mathcal{Y}_D$ denote the set of prototypes of discriminant D.

We associate to each prototype $P = (a, b, c, \bar{q}) \in \mathcal{Y}_D$ the unique algebraic number $\lambda(P) \in K_D$ such that $a\lambda(P)^2 + b\lambda(P) + c = 0$ and $\lambda(P) > 0$. This makes sense because the two roots of $ax^2 + bx + c = 0$ have opposite signs. It is easy to check that the last condition $a + b + c < 0$ is equivalent to $\lambda(P) > 1$.

Notation. Unless said otherwise, a prototype P will always be given by the letters $P = (a, b, c, \bar{q})$, and the letters a, b, c, and $\bar{q}$ will always belong to some implied prototype. The letter λ will usually denote the number $\lambda(P)$ for an implied prototype P. We define

$$(a', b', c', \bar{q}') := \frac{(a, b, c, \bar{q})}{\gcd(a, b, c)},$$

with $\bar{q}' \in \mathbb{Q}/\mathbb{Z}$.

Operations on prototypes. Given a prototype P, define the *next prototype* P^+ by

$$P^+ = \begin{cases} (a, 2a + b, a + b + c, \bar{q}), & \text{if } 4a + 2b + c < 0; \\ (-a - b - c, -2a - b, -a, \bar{q}), & \text{if } 4a + 2b + c > 0. \end{cases}$$

Given a prototype P, define the *previous prototype* P^- by

$$P^- = \begin{cases} (a, -2a + b, a - b + c, \bar{q}), & \text{if } a - b + c < 0; \\ (-c, -b + 2c, -a + b - c, \bar{q}), & \text{if } a - b + c > 0. \end{cases}$$

Its easy to check that P^+ and P^- are actually prototypes of the same discriminant and that

$$(P^+)^- = (P^-)^+ = P.$$

Define an involution t on the set of prototypes of discriminant D by

$$t(a, b, c, \bar{q}) = \begin{cases} (a, -b, c, \bar{q}), & \text{if } a - b + c < 0; \\ (-c, b, -a, \bar{q}), & \text{if } a - b + c > 0. \end{cases}$$

6 Three-cylinder surfaces

We define

$$\Upsilon_D \subset \mathcal{X}_D$$

to be the set of directed surfaces $(X, [\omega], v)$ for which the direction v is periodic, and we let $\Upsilon_D^i \subset \mathcal{P}$ be the locus where the associated cylinder decomposition has i cylinders. The goal of this section is to classify the connected components of Υ_D^3 and define explicit coordinates on each connected component.

Parameterizing three-cylinder surfaces. Given two complex numbers z_i, let $P(z_1, z_2)$ be the parallelogram containing as sides the two segments $[0, z_i]$.

Given positive real numbers x_1, x_2, and x_3 such that

$$x_2 = x_1 + x_3,$$

and complex numbers y_1, y_2, and $y_3 \in \mathbb{H}$, define a surface,

$$S(x_1, x_2, x_3, y_1, y_2, y_3),$$

to be the surface (X, ω) obtained by gluing the three parallelograms $P_i = P(x_i, y_i)$ as in Figure 3. This surface only depends on y_i mod $x_i\mathbb{Z}$. We equip (X, ω) with the homology classes $\alpha_i \in H_1(X; \mathbb{Z})$ and $\gamma_i \in H_1(X, Z(\omega); \mathbb{Z})$ indicated in Figure 3.

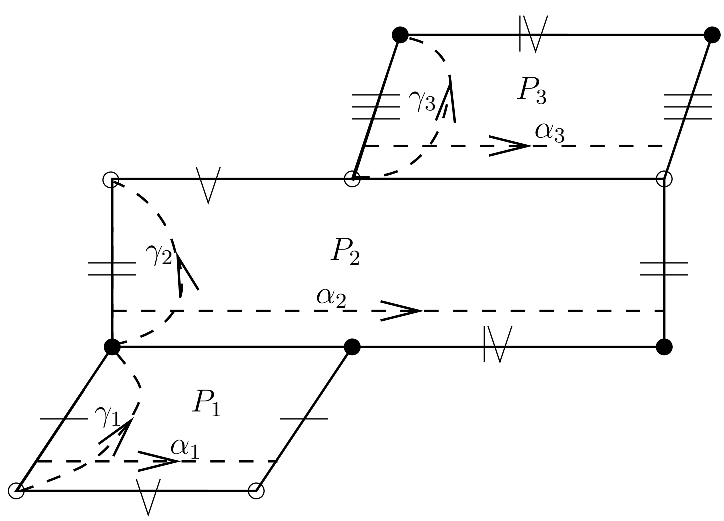

Figure 3: Three-cylinder surface

Define homology classes,

$$\beta_1 = \gamma_1 - \gamma_3, \quad \text{and} \quad \beta_2 = \gamma_2 + \gamma_3.$$

Then

$$(\alpha_1, \alpha_2, \beta_1, \beta_2) \tag{6.1}$$

forms a symplectic basis of $H_1(X;\mathbb{Z})$.

Canonical representation. We now define a canonical representation of every surface in Υ_D^3 as some $S(\mathbf{x}, \mathbf{y})$.

Proposition 6.1. *Each directed surface $(X, [\omega], v) \in \Upsilon_D^3$ is of the form,*

$$(X, [\omega], v) = S(\mathbf{x}, \mathbf{y}) = S(1, \mu, \mu - 1, y_1, y_2, y_3),$$

for some $\mu \in K_D$ with $\mu > 1$ and $N(\mu) < 0$. This representation is unique taking y_i mod $x_i\mathbb{Z}$.

Proof. Since the horizontal direction is periodic, we can put the surface in the form, $(X, [\omega], v) = S(\mathbf{x}, \mathbf{y})$, for some $\mathbf{x}$ and $\mathbf{y}$. The module $M = \langle \alpha_1, \alpha_2 \rangle \subset H_1(X;\mathbb{Z})$ is an $\mathcal{O}_D$-submodule because it is the set of all homology classes with real periods. Thus we have $x_1/x_j \in K_D$ for any i, j since ω is an eigenform.

We claim that $N_{\mathbb{Q}}^{K_D}(x_3/x_1) < 0$. To see this, let $N = H_1(X;\mathbb{Z})/M$, another $\mathcal{O}_D$-module. The intersection pairing on homology gives a perfect pairing of $\mathcal{O}_D$-modules,

$$M \times N \to \mathbb{Z}.$$

The bases

$$(u_1, u_2) = (\alpha_1, \alpha_3) \quad \text{and} \quad (v_1, v_2) = (\beta_1 + \beta_2, \beta_2)$$

are dual bases of M and N respectively. They satisfy $\omega(u_i) > 0$ and $\operatorname{Im}\omega(v_i) > 0$. The claim then follows directly from Theorem 3.5 of [Bai].

We now have either

$$N_{\mathbb{Q}}^{K_D}(x_2/x_1) < 0 \quad \text{or} \quad N_{\mathbb{Q}}^{K_D}(x_3/x_2) < 0, \tag{6.2}$$

but not both. If the second holds, then swap x_1 and x_3 as well as y_1 and y_3, which does not change the surface. We can then assume $N(x_2/x_1) < 0$. Finally divide $(\mathbf{x}, \mathbf{y})$ by x_1 to put the surface in the required form.

This representation is unique because if there were two such representations, then both alternatives in (6.2) would hold, a contradiction. ■

We will always assume that any point in Υ_D^3 is represented by the surface (X, ω) which is the canonical representative given in Proposition 6.1, and X will be equipped with the homology classes α_i, β_i, and γ_i defined above. The class γ_i is really only defined up to adding a multiple of α_i. We just choose any γ_i so that $\alpha_i \cdot \gamma_j = \delta_{ij}$.

Prototypes. We now assign to every $(X,\omega) \in \Upsilon^3_D$ a prototype $P(X,\omega)$.

Let $\mu = \omega(\alpha_2) \in K_D$, and define $\phi_\mu(x) = ax^2 + bx + c$ to be the unique multiple of the minimal polynomial of μ such that $b^2 - 4ac = D$ and $a > 0$. Let T be the matrix of the action of $a\mu$ on $H_1(X;\mathbb{Q})$ in the symplectic basis (6.1). Since T is self-adjoint with respect to the intersection pairing, it is of the form

$$T = \begin{pmatrix} 0 & -c & 0 & q \\ a & -b & -q & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -c & -b \end{pmatrix}. \tag{6.3}$$

We define $P(X,\omega) = (a,b,c,\overline{q})$.

Proposition 6.2. *$P(X,\omega)$ is a well-defined prototype.*

Proof. We have $c < 0$ and $a + b + c < 0$ because $\mu > 1$ and $N^{K_D}_{\mathbb{Q}}(\mu) < 0$ by Proposition 6.1. Since $\mathcal{O}_D \cong \mathbb{Z}[a\mu]$, we must have $\gcd(a,b,c,\bar{q}) = 1$, or else the action of $\mathcal{O}_D$ would extend to an order $\mathcal{O}_E \supset \mathcal{O}_D$.

To see that $P(X,\omega)$ is well-defined, we must check that it is independent of the choice of the classes γ_i. This is straightforward; see [Bai, Theorem 7.21] for the proof. ∎

Coordinates on Υ^3_D. Let $U_P \subset \Upsilon^3_D$ be the set of surfaces with prototype P. We now give coordinates for Υ^3_D by parameterizing each U_P.

Lemma 6.3. *The surface $(X,\omega) = S(1,\mu,\mu-1,y_1,y_2,y_3)$ represents an eigenform in U_P if and only if*

$$\begin{gathered} \mu = \lambda(P), \quad \textit{and} \\ a' y_1 + \frac{c'}{\mu} y_2 + \frac{a'+b'+c'}{\mu-1} y_3 \equiv -q' \pmod{\mathbb{Z}}, \end{gathered} \tag{6.4}$$

where $P = (a,b,c,\overline{q})$.

Proof. First, suppose the equations (6.4) hold. We can then choose γ_i such that $\alpha_i \cdot \gamma_j = \delta_{ij}$ and the following equations hold:

$$\begin{aligned} a\mu\omega(\alpha_2) &= -c\omega(\alpha_1) - b\omega(\alpha_2), \\ a\mu\omega(\beta_1) &= -q\omega(\alpha_2) - c\omega(\beta_2), \quad \text{and} \\ a\mu\omega(\beta_2) &= q\omega(\alpha_1) + a\omega(\beta_1) - b\omega(\beta_2). \end{aligned} \tag{6.5}$$

We then define real multiplication of $\mathcal{O}_D$ on $\mathrm{Jac}(X)$ by

$$a\mu \cdot x = Tx, \tag{6.6}$$

where T is the matrix of (6.3). This defines an action of $\mathcal{O}_D$ using $\mathcal{O}_D \cong \mathbb{Z}[a\mu]$, By the above equations (6.5), $\omega(\lambda \cdot x) = \lambda^{(1)}\omega(x)$ for all $x \in H_1(X;\mathbb{Z})$. So by Proposition 4.12, this exhibits (X,ω) as an eigenform with prototype P.

Conversely, suppose (X,ω) is an eigenform with prototype P. The real multiplication is defined by (6.6) with T as in (6.3). Since (X,ω) is an eigenform, the equations (6.5) hold mod $\mathbb{Z}$, and (6.4) follows. ∎

Define

$$\mathfrak{U}_P = \left\{ (y_1, y_2, y_3) \in \mathbb{H}/\mathbb{Z} \times \mathbb{H}/\lambda\mathbb{Z} \times \mathbb{H}/(\lambda-1)\mathbb{Z} : \right.$$
$$\left. a'y_1 + \frac{c'}{\lambda}y_2 + \frac{a'+b'+c'}{\lambda-1}y_3 \equiv -q' \pmod{\mathbb{Z}} \right\}, \quad (6.7)$$

and define $\phi_P \colon \mathfrak{U}_P \to U_P$ by

$$\phi_P(y_1, y_2, y_3) = S(1, \lambda, \lambda - 1, y_1, y_2, y_3).$$

Theorem 6.4. *The map $\phi_P \colon \mathfrak{U}_P \to U_P$ is a biholomorphic isomorphism.*

Proof. By Lemma 6.3, ϕ_P has image in U_P and is surjective. By Proposition 6.1, ϕ_P is injective. Finally, ϕ_P is locally biholomorphic by Theorem 3.1. ∎

Corollary 6.5. *The connected components of Υ_D^3 correspond bijectively to prototypes of discriminant D.*

Proof. By Theorem 6.4, U_P is connected, so $P \mapsto U_P$ defines the required bijection. ∎

Covering of U_P. We will have use for the following covering of U_P. Define

$$\widehat{U}_P = \mathbb{H}/\frac{a'}{\gcd(a', c')}\lambda\mathbb{Z} \times \mathbb{H}/\frac{a'}{\gcd(a', a'+b'+c')}(\lambda-1)\mathbb{Z},$$

which is an $a'/\gcd(a', c')\gcd(a', a'+b'+c')$-fold covering of U_P via

$$g(y_2, y_3) = \phi_P\left(-\frac{c'}{a'\lambda}y_2 - \frac{a'+b'+c'}{a'(\lambda-1)}y_3 - \frac{q'}{a'}, y_2, y_3\right). \quad (6.8)$$

7 Compactification of X_D

Let Y_D be the normalization as an algebraic variety of $\overline{X}_D$, the closure of X_D in $\mathbb{P}\Omega\overline{\mathcal{M}}_2$. We showed in [Bai]:

Theorem 7.1. *Y_D is a compact, complex projective orbifold.*

In this section, we summarize more of the properties of the compactification Y_D which we proved in [Bai]. In particular, we discuss the combinatorics of curves in ∂X_D, and we give explicit local coordinates around points in ∂X_D.

Curves in $\mathbb{P}\Omega\overline{\mathcal{M}}_2$. We start by discussing some curves in $\mathbb{P}\Omega\overline{\mathcal{M}}_2$ which will be covered by curves in ∂X_D.

Given $\lambda > 1$ define $\mathcal{C}_\lambda \subset \mathbb{P}\Omega\overline{\mathcal{M}}_2$ to be the closure of the locus of stable Abelian differentials $(X, [\omega])$ with two separating nodes such that the ratio of the residues of ω at the nodes is $\pm\lambda^{\pm 1}$.

Let $c_\lambda \in \mathbb{P}\Omega\overline{\mathcal{M}}_2$ be the point representing a stable Abelian differential with three nonseparating nodes having residues 1, λ, and $\lambda - 1$.

Let $p_\lambda \in \mathbb{P}\Omega\overline{\mathcal{M}}_2$ be the differential (X, ω) obtained by joining infinite cylinders of circumference 1 and λ at one point to form a node. The form (X, ω) has one separating node where ω has residue zero, and two nonseparating nodes where ω has residues 1 and λ.

Let $w_\lambda \in \mathbb{P}\Omega\overline{\mathcal{M}}_2$ be the unique form (X, ω) having a double zero and two nonseparating nodes having residues 1 and λ. The form (X, ω) can be formed as follows. Start with an infinite cylinder C_1 of circumference λ. Cut C_1 along a segment of length 1 and identify opposite ends of the segment to form a "figure eight" as in Figure 4. Then glue two half-infinite cylinders of circumference 1 to the boundary of the figure eight as shown in Figure 4.

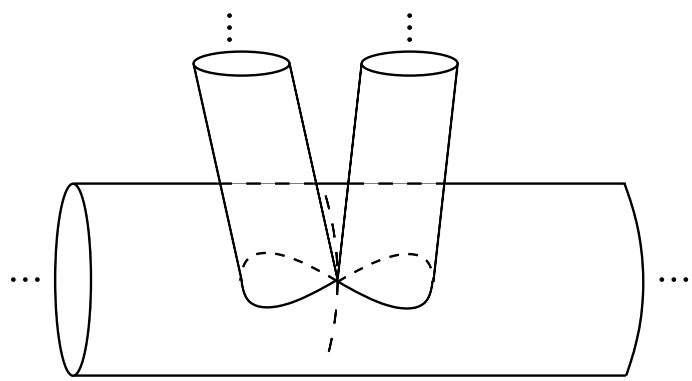

Figure 4: w_λ

From Proposition 6.9 of [Bai], we have

Proposition 7.2. *The curve $\mathcal{C}_\lambda$ is a rational curve containing the points c_λ, $c_{\lambda+1}$, w_λ, and p_λ. Every other point of $\mathcal{C}_\lambda$ represents a stable differential with two nonseparating nodes and two simple zeros.*

We normalize every differential (X, ω) in $\mathcal{C}_\lambda$ so that ω has residues 1 and λ at two of the nodes.

Proposition 7.3. *With this normalization, the horizontal foliation of every $(X, \omega) \in \mathcal{C}_\lambda$ is periodic.*

Proof. Each of the four half-infinite cylinders of (X, ω) is bounded by a union of saddle connections. None of these saddle connections bounds two cylinders, or else (X, ω) would be a one-point connected sum of two infinite cylinders and we would be done. Since ω has two zeros, counting multiplicity, this means that every separatrix of (X, ω) starts and ends at a zero. But this implies (X, ω) is periodic. ■

We now define a meromorphic quadratic differential q_λ on $\mathcal{C}_\lambda$ as we did on the leaves of $\mathcal{A}_D$ in §4. Let $\mathcal{C}_\lambda(1,1) = \mathcal{C}_\lambda \setminus \{c_\lambda, c_{\lambda+1}, p_\lambda, w_\lambda\}$. Define a

multivalued holomorphic function f_λ on $\mathcal{C}_\lambda(1,1)$ by

$$f_\lambda(X,\omega) = \int_p^q \omega,$$

where p and q are the zeros of ω, and define $q_\lambda = (\partial f_\lambda)^2$.

Proposition 7.4. *The differential q_λ has double poles at c_λ and $c_{\lambda+1}$, a simple pole at p_λ, a simple zero at w_λ, and is holomorphic and nonzero on $\mathcal{C}_\lambda(1,1)$. The horizontal foliation of q_λ is periodic, and the spine of q_λ coincides with the locus of two-cylinder surfaces.*

Proof. Any surface (X,ω) sufficiently close to c_λ or $c_{\lambda+1}$ is a three cylinder surface with one finite area cylinder C. Define a loop on $\mathcal{C}_\lambda$ by cutting (X,ω) along a closed geodesic of C and regluing after a rotation. This is a loop of length λ or $\lambda+1$ contained in the horizontal foliation of q_λ. Thus $(\mathcal{C}_\lambda, q_\lambda)$ has half-infinite cylinders around c_λ and $c_{\lambda+1}$, so q_λ has double poles there.

The other zeros and poles are located just as for the quadratic differentials we defined along $T\mathcal{A}_D$ (see §10 of [Bai]).

Points along the separatrices emanating from p_λ and w_λ are obtained by performing a connected sum on p_λ or splitting the double zero of w_λ along a horizontal segment, which does not create a new cylinder. Thus the separatrices are contained in the two-cylinder locus, an embedded graph on $\mathcal{C}_\lambda$. Thus the separatrices are saddle connections, and the horizontal foliation of q_λ is periodic.

By period coordinates, there is a tubular neighborhood U of the spine such that ∂U consists of three cylinder surfaces. We then have $a_t \cdot \partial U \to \{c_\lambda \cup c_{\lambda+1}\}$ as $t \to \infty$, so $\cup a_t \cdot U$ covers $\mathcal{C}_\lambda \setminus \{c_\lambda \cup c_{\lambda+1}\}$. It follows that the complement of the spine consists of three-cylinder surfaces as claimed. ■

The surface $(\mathcal{C}_\lambda, q_\lambda)$ with its horizontal foliation is shown in Figure 5 with the spine indicated by a solid line. The singular points of the foliation with three or one prong are w_λ and p_λ respectively.

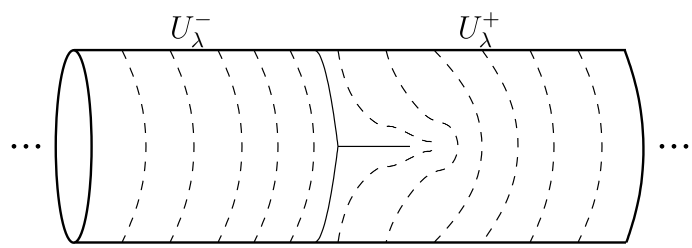

Figure 5: $\mathcal{C}_\lambda$ with the quadratic differential q_λ

The locus of three-cylinder surfaces in $\mathcal{C}_\lambda$ consists of two connected components. We refer to the component containing $c_{\lambda+1}$ as U_λ^- and we call the other component U_λ^+.

Curves in ∂X_D. In [Bai], we defined for every prototype $P = (a, b, c, \overline{q})$ a curve $C_P \subset \partial X_D$ and a point $c_P \in \partial X_D$. The following properties of Y_D are proved in §8 of that paper:

Theorem 7.5. *Y_D has the following properties:*

- *$\partial X_D = \bigcup_P C_P$, where the union is over all prototypes of discriminant D.*
- *C_P is a rational curve which intersects C_{P+} at c_P and intersects C_{P-} at c_{P-}. There are no other intersections between the curves C_P.*
- *The restriction of the natural morphism $\pi\colon Y_D \to \mathbb{P}\Omega\overline{\mathcal{M}}_2$ realizes C_P as a $\gcd(a', c')$-fold branched cover of $\mathcal{C}_{\lambda(P)}$ with $\pi(c_P) = c_{\lambda(P)}$ and $\pi(c_{P-}) = c_{\lambda(P)+1}$. The restriction $\pi|_{C_P}$ is ramified to order $\gcd(a', c')$ at these two points and is elsewhere unramified.*
- *The points c_P are cyclic quotient singularities of Y_D of order*
$$m_P = \frac{a'}{\gcd(a', c')\gcd(a', b' + c')}, \tag{7.1}$$
and are the only singular points in ∂X_D.
- *The curves $\overline{P}_D$ and $\overline{W}_D$ intersect C_P transversely in $\gcd(a', c')$ points each and are disjoint from the points c_Q.*
- *The involution τ of X_D extends to an involution τ of Y_D which sends C_P to $C_{t(P)}$.*

It follows from this theorem that ∂X_D consists of finitely many chains of rational curves C_P. The connected components of ∂X_D are naturally in bijection with the cusps of X_D.

We equip C_P with a quadratic differential q_P defined by,

$$q_P = (\pi|_{C_P})^* q_{\lambda(P)}.$$

The locus of three-cylinder surfaces in C_P is the complement of the spine of q_P by Proposition 7.4. We define $U_P^- \subset C_P$ to be the component of the three-cylinder locus containing c_P, and we define $U_{P-}^+ \subset C_P$ to be the component containing c_{P-}. Let $S_P \subset C_P$ be the spine of q_P, the two-cylinder locus.

Define
$$\mathcal{U}_P = U_P \cup U_P^+ \cup U_P^- \cup c_P \subset Y_D^\sigma,$$
where σ is the cusp of X_D corresponding to P. The set $\mathcal{U}_P$ is a neighborhood of c_P.

Local coverings. Given a cusp σ of X_D, we defined the covering $X_D^\sigma \to X_D$ which is injective on a neighborhood $\mathfrak{N}_r$ of the cusp of X_D^σ. Let $\overline{\mathfrak{N}}_r$ be the closure of $\mathfrak{N}_r$ in Y_D, and define Y_D^σ to be the complex orbifold obtained by gluing $\overline{\mathfrak{N}}_r$ to X_D^σ. Then Y_D^σ is a complex orbifold consisting of X_D^σ together with a single chain of rational curves, and the natural map $Y_D^\sigma \to Y_D$ is locally biholomorphic.

Each surface $(X, [\omega]) \in Y_D^\sigma$ has a canonical horizontal direction, and we can normalize ω (up to real multiple) so that the horizontal direction of ω coincides with this canonical direction.

We then obtain an action of the upper-triangular subgroup $B \subset \mathrm{SL}_2\mathbb{R}$ on Y_D^σ coming from the action of $\mathrm{GL}_2^+\mathbb{R}$ on $\Omega\overline{\mathcal{M}}_2$. This action fixes the points c_P and fixes S_P pointwise.

Type one coordinates. We now describe local coordinates around points of $C_P \setminus \{c_P \cup c_{P^-}\}$ which will be used in the proof of Lemma 8.3.

Let $W \subset \mathcal{C}_\lambda \setminus \{c_\lambda, c_{\lambda+1}\}$ be a simply connected domain. For each $z \in W$, write (X_z, ω_z) for the associated stable differential, normalized as before so that the residues at the nonseparating nodes are 1 and λ. Write $N = \{n_1, n_2\} \subset X_z$ for the set of nonseparating nodes with n_1 being the residue 1 node. Choose homology classes

$$\alpha_i \in H_1(X_z \setminus N_z; \mathbb{Z}) \quad \text{and} \quad \beta_i \in H_1(X_z, N_z; \mathbb{Z})$$

so that α_i goes around n_i, and $\alpha_i \cdot \beta_j = \delta_{ij}$. Choose these classes consistently so that they are parallel with respect to the Gauss-Manin connection.

We can plumb the nodes n_i as described in §3 to obtain a new surface (X'_z, ω'_z). The homology class β_i becomes a class in $H_1(X'_z; \mathbb{Z})$ which we will continue to call β_i. It is well defined up to adding a multiple of α_i. If w_i is sufficiently large, then there is a unique way to plumb the node n_i so that

$$e^{2\pi i \omega'_z(\beta_i)/\omega'_z(\alpha_i)} = w_i. \tag{7.2}$$

Let $\mathrm{Plumb}(z, w_1, w_2)$ be the result of plumbing (X_z, ω_z) so that (7.2) holds. If one of the w_i is zero, the corresponding node of (X_z, ω_z) remains a node of $\mathrm{Plumb}(z, w_1, w_2)$.

Let $U \subset C_P$ be a component of $\pi^{-1}(W)$. Define $f\colon U \times V \to Y_D^\sigma$ by

$$f(u, v) = \mathrm{Plumb}(\pi(u), v^{-c'/\gcd(a',c')}, r v^{a'/\gcd(a',c')}),$$

where r satisfies

$$r^{-c'} = e^{2\pi i q'}, \tag{7.3}$$

and $V \subset \mathbb{C}$ is a neighborhood of 0 small enough that f is defined on $U \times V$.

From Theorem 7.22 and Proposition 8.1 of [Bai], we have:

Theorem 7.6. *For some choice of r satisfying (7.3), f is biholomorphic onto its image in Y_D^σ, a neighborhood of $U \subset C_P$.*

Remark. The choice of r in (7.3) is equivalent to the choice of the component U of $\pi^{-1}(W)$.

It follows that f defines coordinates (u, v) around points in U. We will call these *type one coordinates.*

Type two coordinates. We now describe local coordinates around the points $c_P \in Y_D$. Define $R_\lambda \colon \Delta^3 \to \mathbb{P}\Omega\overline{\mathcal{M}}_2$ by

$$R_\lambda(w_1, w_2, w_3) = S\left(1, \lambda, \lambda - 1, \frac{1}{2\pi i}\log w_1, \frac{\lambda}{2\pi i}\log w_2, \frac{\lambda-1}{2\pi i}\log w_3\right),$$

where if $w_i = 0$, then the corresponding cylinder is instead a node. Define $g_P \colon \Delta^2 \to \mathbb{P}\Omega\overline{\mathcal{M}}_2$ by

$$g_P(u, v) = R_{\lambda(P)}\Big(e^{2\pi i q'/a'} u^{-c'/\gcd(a',c')} v^{-(a'+b'+c')/\gcd(a',a'+b'+c')}, \\ u^{a'/\gcd(a',c')}, v^{a'/\gcd(a',a'+b'+c')}\Big).$$

Given relatively prime $m, s \in \mathbb{Z}$ with $m > 0$, let θ_m be an m-th root of unity, and let $\Gamma_{m,s}$ be the group of automorphisms of Δ^2 generated by

$$(z, w) \mapsto (\theta_m z, \theta_m^s w).$$

From Theorem 7.27 and Proposition 8.3 of [Bai], we have

Theorem 7.7. *g_P lifts to a holomorphic map $f \colon \Delta^2 \to Y_D^\sigma$, and f descends to a map $\widetilde{f} \colon \Delta^2/\Gamma_{m,s} \to Y_D^\sigma$, where $m = m_P$ defined in (7.1). The map f is biholomorphic onto its image $\mathcal{U}_P$, a neighborhood of c_P. Furthermore,*

$$f^{-1}(C_P) = \{(u, v) \in \Delta^2 \colon u = 0\}$$
$$f^{-1}(C_{P^+}) = \{(u, v) \in \Delta^2 \colon v = 0\}.$$

We call the coordinates (u, v) induced by f on the neighborhood $\mathcal{U}_P$ of c_P *type two coordinates.*

Singularities of $\mathcal{F}_D$. We finish this section by discussing the geometry of the foliation $\mathcal{F}_D$ near ∂X_D.

Proposition 7.8. *The foliation $\mathcal{F}_D$ extends to a foliation of*

$$Y_D \setminus \bigcup_{P \in \mathcal{Y}_D} (c_P \cup S_P) \tag{7.4}$$

and does not extend to any larger set.

Proof. We work in a local covering Y_D^σ where the leaves of $\mathcal{F}_D$ are the orbits of B. Since the B-action is free on the locus (7.4), its orbits there are the required extension of $\mathcal{F}_D$.

In the neighborhood $\mathcal{U}_P$ of c_P, the horizontal foliation of every surface (X,ω) has three cylinders, so $a_t \cdot (X,\omega) \to c_P$. Thus every leaf of $\mathcal{F}_D$ near c_P passes through c_P, which is impossible if $\mathcal{F}_D$ can be extended to a foliation.

Suppose $\mathcal{F}_D$ can be extended over a neighborhood of $p \in S_P$. Choose type one coordinates (u,v) around p so that $p=(0,0)$. In these coordinates $\{0\}\times\Delta_\epsilon$ is contained in a leaf of $\mathcal{F}_D$ transverse to C_P. The subsets U_P^- and $U_{P^-}^+$ of C_P are also contained in leaves of $\mathcal{F}_D$. Thus C_P is a leaf of $\mathcal{F}_D$ near p, a contradiction because then $\mathcal{F}_D$ would have two leaves meeting at p. ■

8 $\mathcal{F}_D$ as a current on Y_D

We saw in Proposition 7.8 that $\mathcal{F}_D$ doesn't extend to a foliation of Y_D. Nevertheless, we will show in this section that the closed current on X_D defined by the measured foliation $\mathcal{F}_D$ does extend to Y_D:

Theorem 8.1. *Integration of smooth* 2*-forms on* Y_D *over* $\mathcal{F}_D$ *defines a closed* 2*-current on* Y_D*.*

We will then use this to interpret $\operatorname{vol} E_D(1,1)$ as an intersection of classes in $H^2(Y_D;\mathbb{R})$.

Poincaré growth. Let $\overline{X}$ be a compact, complex orbifold, and let $X\subset\overline{X}$ be the complement of a divisor D. Suppose D is covered by coordinate charts of the form Δ^n/G, where

- Each transformation $g\in G$ is of the form,
$$g(z_1,\ldots,z_n)=(\theta_1 z_1,\ldots,\theta_n z_n),$$
for some roots of unity θ_i;
- $D\cap\Delta$ is a union of the coordinate axes $z_1=0,\ldots,z_r=0$ for $1\le r\le n$.

In such a chart Δ^n/G, we have $\Delta^n\cap X=(\Delta^*)^r\times\Delta^{n-r}$. We give $\Delta^n\cap X$ a metric ρ by putting the Poincaré metric,
$$ds^2=\frac{|dz|^2}{|z|^2(\log|z|)^2},$$
on the Δ^* factors, putting the Euclidean metric $|dz|^2$ on the Δ factors, and defining ρ to be the product metric. Following Mumford [Mum77], we say that a form ω on X has *Poincaré growth* if there is a covering of D by charts U_α/G_α with metrics ρ_α of this form such that for each α,
$$\|\omega\|_{\rho_\alpha}\prec 1.$$

Let ω_i be the 2-form on X_D covered by the forms,

$$\widetilde{\omega}_i = \frac{1}{2\pi}\frac{dx_i \wedge dy_i}{y_i^2},$$

on $\mathbb{H} \times \mathbb{H}$.

Proposition 8.2. *The forms ω_i have Poincaré growth. Furthermore, there are smooth 1-forms η_i on X_D with Poincaré growth such that $\omega_i - d\eta_i$ are smooth 2-forms on Y_D. Integration against ω_i defines a closed 2-current on Y_D.*

Proof. We showed in [Bai, Proposition 2.8] that ω_i is the Chern form of a metric h_i on a line bundle $Q^i Y_D$ over Y_D, and we showed in [Bai, Theorem 9.8] that h_i is what Mumford calls a good metric in [Mum77]. It follows from [Mum77, Theorem 1.4] that the ω_i have Poincaré growth. The second statement follows from the proof of the same theorem, and the last statement follows from Propositions 1.1 and 1.2 of [Mum77]. ■

Bounded norm. In order to prove Theorem 8.1, we need to bound the norm $\|\omega\|_{\mathcal{F}_D}$ for smooth forms ω on Y_D. In fact, we will need these bounds for forms which just have Poincaré growth.

Lemma 8.3. *For any p-form ω on X_D with Poincaré growth, $\|\omega\|_{\mathcal{F}_D}$ is bounded.*

Proof. Let $U \subset Y_D$ be an open set meeting ∂X_D with the coordinates (u, v) defined in §7. Let $K \subset U$ be compact. Let ρ be the metric on U as defined above. Since $\|\omega\|_\rho$ is bounded on K, it suffices to show that $\|v\|_\rho / \|v\|_{\mathcal{F}_D}$ is bounded on K for any vector field v tangent to $\mathcal{F}_D$ (where $\|v\|_{\mathcal{F}_D}$ denotes the norm of v with respect to the hyperbolic metric on leaves of $\mathcal{F}_D$). Since $\mathcal{F}_D$ is one-dimensional, it suffices to show $\|v\|_\rho$ is bounded for a single vector field v tangent to $\mathcal{F}_D$ with unit $\mathcal{F}_D$-norm. There are two cases, depending on whether or not U is a type one or type two coordinate chart.

First suppose U is a type two coordinate chart around c_P in Y_D. If U is small enough, we can regard it as a subset of U_P from §6. By Theorem 6.4, the universal cover $\widetilde{U}_P$ is isomorphic to $\mathbb{H} \times \mathbb{H}$ with coordinates (y_2, y_3) from §6. The (u, v)-coordinates are related to the (y_2, y_3)-coordinates by

$$u = e^{ic_2 y_2} \quad \text{and} \quad v = e^{ic_3 y_3},$$

for positive real constants c_2 and c_3. The action of the diagonal group $A \subset \mathrm{SL}_2\mathbb{R}$ on $\widetilde{U}_P$ is given by

$$a_t \cdot (u_2 + iv_2, u_3 + iv_3) = (u_2 + ie^t v_2, u_3 + ie^t v_3),$$

where $y_j = u_j + iv_j$. Thus

$$v = \operatorname{Im} y_2 \frac{\partial}{\partial y_2} + \operatorname{Im} y_3 \frac{\partial}{\partial y_3}$$

is a unit length tangent vector field to the lift of $\mathcal{F}_D$ to $\widetilde{U}_P$. In the (u,v)-coordinates,

$$v = -iu \log|u| \frac{\partial}{\partial u} - iv \log|v| \frac{\partial}{\partial v}.$$

This is a unit length vector field tangent to $\mathcal{F}_D$ with bounded ρ-length.

The case when U is a type one coordinate chart is similar. For some function f, the vector field,

$$v = f(u,v)\frac{\partial}{\partial u} - iv \log|v| \frac{\partial}{\partial v},$$

is a unit length vector field tangent to $\mathcal{F}_D$. The function f is continuous because v is tangent to the flow on Y_D generated by A. Therefore $\|v\|_\rho$ is bounded on K. ∎

Corollary 8.4. *For any 2-form ω on X with Poincaré growth,*

$$\int_{\mathcal{F}_D} |\omega| < \infty. \tag{8.1}$$

In particular, $\mathcal{F}_D$ defines a current on Y_D.

Proof. The first statement follows directly from Lemma 8.3 since μ_D has finite volume by Theorem 4.6. Since any smooth form on Y_D has Poincaré growth, (8.1) holds in particular for smooth forms. For $\mathcal{F}_D$ to define a current, it suffices to show that if $\omega_n \to 0$ is a uniformly convergent sequence of forms on Y_D, then $\int_{\mathcal{F}_D} \omega_n \to 0$. This is easily seen from the proof of Lemma 8.3 using the fact that $\|\omega_n\|_\rho \to 0$ in each chart U. ∎

Cusp neighborhoods. The proof that $\mathcal{F}_D$ is closed, will require certain well-behaved neighborhoods of the cusps of X_D.

Lemma 8.5. *For any $\epsilon > 0$, there is a closed, H-invariant neighborhood $W \subset X_D$ of the cusps of X_D such that* $\operatorname{vol} W < \epsilon$ *and ∂W is a submanifold of X_D transverse to $\mathcal{F}_D$.*

Proof. It suffices to construct a neighborhood of the cusp at infinity in the covering X_D^σ with the required properties. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be smooth and increasing with $\rho(x) = 0$ for $x < 1/2$ and $\rho(x) = x$ for $x > 1$. Recall that each point in X_D^σ represents an eigenform (X,ω) together with a choice of horizontal direction. Normalize each (X,ω) to have unit area. Define a smooth function f on X_D^σ by

$$f(X,\omega) = \sum_i \rho(\mathrm{height}(C_i)),$$

where the sum is over the horizontal cylinders of X_D (f is zero if there are no horizontal cylinders). The bump function ρ is there to make f smooth; otherwise f would just be continuous.

Let $W_\ell = f^{-1}[\ell,\infty)$. The function f is H-invariant, so W_ℓ is as well. We have $f(x) \to \infty$ as x approaches the cusp of X_D^σ because surfaces develop tall

cylinders as they approach the cusp (see Theorem 5.5 and Proposition 7.1 of [Bai]). Thus W_ℓ is a neighborhood of the cusp.

If $f(X,\omega) > 0$, then $\frac{d}{dt} f(a_t \cdot (X,\omega)) > 0$. It follows that ℓ is a regular value of f, so ∂W_ℓ is smooth if $\ell > 0$. Also if $\ell > 0$, then ∂W_ℓ is transverse to $\mathcal{F}_D$.

Since X_D has finite volume, the cusp of X_D^σ does as well, so $\operatorname{vol} W_\ell \to 0$ as $\ell \to \infty$. ■

Given W as in Lemma 8.5, let $\mathcal{W}$ be the measured foliation of ∂W induced by $\mathcal{F}_D$. Give the leaves of $\mathcal{W}$ their arclength measure as horocycles in $\mathbb{H}$, and let ν be the measure on ∂W which is the product of the leafwise and transverse measures.

Lemma 8.6. *We have*

$$\operatorname{vol} \partial W = \operatorname{vol} W$$

with respect to the measures ν and μ_D.

Proof. This follows directly from the fact that the length of the horocycle in $\mathbb{H}$ joining i to $i+t$ is equal to the area of the region in $\mathbb{H}$ lying above this horocycle. ■

Proof of Theorem 8.1. After Corollary 8.4, it remains to show that the current defined by integration over $\mathcal{F}_D$ is closed. Let η be a smooth 1-form on Y_D. We need to show $\int_{\mathcal{F}_D} d\eta = 0$.

Since η is smooth, η and $d\eta$ have Poincaré growth, so $\|\eta\|_{\mathcal{F}_D}$ and $\|\eta\|_{\mathcal{F}_D}$ are bounded. Given $\epsilon > 0$, let $W \subset X_D$ be the neighborhood of the cusps constructed in Lemma 8.5 with $\operatorname{vol} W < \epsilon$. By Stokes' Theorem and Lemma 8.6,

$$\begin{aligned}\int_{\mathcal{F}_D} d\eta &= \int_{\mathcal{F}_D|W} d\eta + \int_{\mathcal{W}} \eta \\ &\leq \epsilon(\|\eta\|^\infty_{\mathcal{F}_D} + \|d\eta\|^\infty_{\mathcal{F}_D})\end{aligned}$$

where the last integral is over the measured foliation of $\mathcal{W}$ of ∂W defined above. Thus $\int_{\mathcal{F}_D} d\eta = 0$. ■

Cohomological interpretation of $\operatorname{vol} E_D(1,1)$**.** Let $[\overline{\mathcal{F}}_D], [\omega_i] \in H^2(Y_D;\mathbb{R})$ be the cohomology classes defined by the respective closed currents.

Theorem 8.7. *We have*

$$\operatorname{vol} E_D(1,1) = 2\pi[\omega_1] \cdot [\overline{\mathcal{F}}_D],$$

where the pairing is the intersection pairing on $H^2(Y_D;\mathbb{R})$.

Proof. By Proposition 8.2, there is a smooth 1-form η on X_D with Poincaré growth such that

$$\alpha = \omega_1 - d\eta$$

is a smooth 2-form on Y_D. We have,

$$\begin{aligned}
[\omega_1] \cdot [\overline{\mathcal{F}}_D] &= \int_{\mathcal{F}_D} \alpha \\
&= \int_{\mathcal{F}_D} \omega_1 - \int_{\mathcal{F}_D} d\eta \\
&= \int_{\mathcal{F}_D} \omega_1 \\
&= \frac{1}{2\pi} \operatorname{vol} E_D(1,1).
\end{aligned}$$

■

9 Intersection with closed leaves

In this section, we will relate the foliation $\mathcal{F}_D$ in a neighborhood of P_D and W_D with the canonical foliations of the tangent bundles of these curves. This, together with Theorem 2.3 will imply that the classes $[\overline{P}_D]$ and $[\overline{W}_D]$ have trivial intersection with $[\overline{\mathcal{F}}_D]$.

Bundles over Y_D. The line bundle ΩX_D over X_D defined in §4 extends to a line bundle ΩY_D over Y_D whose fiber over $(X, [\omega])$ is the line in $\Omega(X)$ determined by $[\omega]$. Let $QY_D = (\Omega Y_D)^2$, the line bundle whose fiber over $(X, [\omega])$ is the space QX of quadratic differentials on X which are multiples of ω^2. The L^1-metric on QY_D is the Hermitian metric which assigns to q norm $\int_X |q|$.

Cotangent bundles. For the rest of this section, C will denote either of the curves P_D or W_D. For any curve $E \subset Y_D$, we let QE denote the restriction of QY_D to E. We start by sketching a proof of the following well-known fact from Teichmüller theory:

Proposition 9.1. *There is a natural isomorphism,*

$$QC \to T^*C, \tag{9.1}$$

which identifies the L^1-metric on QC with twice the hyperbolic metric on T^*C.

Proof. Suppose $C = W_D$. Given $(X, [\omega]) \in W_D$, the quadratic differential ω^2 defines via Teichmüller theory a cotangent vector τ to $\mathcal{M}_2$ at X. Pulling back τ by the immersion $W_D \to \mathcal{M}_2$ defines a cotangent vector σ to W_D at $(X, [\omega])$. This defines the map (9.1).

Consider the unit speed geodesic $(X_t, \omega_t) = a_t \cdot (X, \omega)$ on W_D. The induced Teichmüller map $f_t \colon X \to X_t$ has complex dilatation,

$$\mu_t = \frac{e^{-t/2} - e^{t/2}}{e^{-t/2} + e^{t/2}} \nu,$$

where $\nu = \overline{\omega}/\omega$. We have

$$\frac{d}{dt}\mu_t = -\frac{1}{2}\nu,$$

so ν represents a tangent vector to W_D of hyperbolic norm 2. The pairing with σ is given by

$$\langle \sigma, \nu \rangle = \int \frac{\overline{\omega}}{\omega}\omega^2 = \int |\omega^2| = \|\omega^2\|,$$

so the hyperbolic norm of σ is half the L^1 norm.

The case where $C = P_D$ is identical, except that we use the immersion $P_D \to \mathcal{M}_1 \times \mathcal{M}_1$ instead of $\mathcal{M}_2$. ■

Lemma 9.2. *The natural isomorphism* (9.1) *extends to an isomorphism,*

$$Q\overline{C} \to T^*\overline{C}(D),$$

where the divisor D is the sum of the cusps of C.

Proof. Given a cusp c, we need to construct a nonzero holomorphic section of $Q\overline{C}$ on a neighborhood of c whose associated section of $T^*\overline{C}$ has a simple pole at c. Let $(X, \omega) \in C$ with $a_t \cdot (X, \omega) \to c$ at $t \to \infty$. Let h_r be a generator of the subgroup of the stabilizer $\mathrm{Stab}(X, \omega) \subset \mathrm{SL}_2\mathbb{R}$ which stabilizes (X, ω). Define a holomorphic map $f\colon \Delta \to C$ by $f(w) = (X_w, \omega_w)$, where if $w \neq 0$, then

$$(X_w, \omega_w) = \begin{pmatrix} 1 & \dfrac{r}{2\pi}\arg w \\ 0 & \dfrac{r}{2\pi}\log\dfrac{1}{|w|} \end{pmatrix} \cdot (X, \omega),$$

and (X_0, ω_0) is the unique limiting stable Abelian differential, which is visibly nonzero. The restriction of f to some Δ_ϵ is a conformal isomorphism onto some neighborhood of c.

We have a nonzero holomorphic section $w \mapsto \omega_w^2$ of $Q\overline{C}$ over Δ_ϵ. The associated holomorphic 1-form τ on Δ_ϵ^* has hyperbolic norm,

$$\|\tau\| = \frac{1}{2}\int_X |\omega_w|^2 \asymp -\log|w|,$$

with respect to the hyperbolic metric on Δ_ϵ^*. Since a nonzero holomorphic section of $T^*\Delta_\epsilon$ has norm comparable to $-|w|\log|w|$, τ has a simple pole at 0. ■

Foliated tubular neighborhoods. Let $U \subset \mathbb{P}\Omega\overline{\mathcal{M}}_2$ be a tubular neighborhood of $\overline{C}$ which is small enough that each point of $U \setminus \overline{C}$ represents a surface with a unique shortest saddle connection (up to the hyperelliptic involution) connecting distinct zeros.

Define a map $\Phi\colon U \to (Q\overline{C})^*$ as follows. Given $(X, \omega) \in U$, let $I \subset X$ be the unique shortest saddle connection connecting distinct zeros, and let (Y, η)

be the surface obtained by collapsing I. Identify QY with QX by the unique linear map T sending η^2 to ω^2. Define $S \in (QY_D)^*$ by

$$S(q) = \left(\int_I \sqrt{T(q)} \right)^2,$$

and define $\Phi(X, \omega) = (Y, S)$.

We observed in the proof of [Bai, Theorem 12.2]:

Proposition 9.3. *In the case where $C = P_D$, the map Φ is an isomorphism between U and a neighborhood of the zero section in $(Q\overline{P}_D)^*$. In the case where $C = W_D$, there is a tubular neighborhood $V \subset U$ which is a threefold branched covering by Φ of a neighborhood of the zero section in $(Q\overline{W}_D)^*$, branched over $\overline{W}_D$.*

The proof is essentially immediate from the fact that given a short segment $I \subset \mathbb{C}$, there are three ways to split a double zero along I and one way to perform a connected sum along I.

Proposition 9.4. *The map Φ sends the foliation $\mathcal{F}_D$ to $2i$ times the canonical foliation of TC.*

Proof. Given $(X, \omega) \in C$, identify the universal cover $\widetilde{C}$ with $\mathbb{H}$ by $z \mapsto (X_z, \omega_z)$, where

$$(X_z, \omega_z) = \begin{pmatrix} 1 & \operatorname{Re} z \\ 0 & \operatorname{Im} z \end{pmatrix} \cdot (X, \omega).$$

By the proof of Proposition 9.1, the Beltrami differential $\nu_z = \overline{\omega}_z/\omega_z$ on X_z represents the tangent at z to the geodesic $\gamma(s) = e^s z$ at $s = t$, that is, the vector

$$v_z = -2iy \frac{\partial}{\partial z},$$

at z.

Now restrict to the case where $C = P_D$. Let $U \subset \mathbb{H}$ be a small neighborhood of i. Given $w \in \mathbb{C} \setminus \{0\}$ let $I_w \in \mathbb{C}$ be the segment joining $\sqrt{w}/2$ and $-\sqrt{w}/2$. Choose some ϵ small enough that for every $w \in \Delta_\epsilon$ and $z \in U$, we can form the connected sum of (X_z, ω_z) along I_w. We parameterize a neighborhood of (X, ω) in Y_D by $f\colon U \times \Delta_\epsilon \to Y_D$, where

$$f(z, w) = \operatorname{Sum}((X_z, \omega_z), I_w). \tag{9.2}$$

In these (z, w)-coordinates, we have $\Phi(z, w) = (X_z, S_w)$, where $S_w \in (QX)^*$ is characterized by $S(\omega_z^2) = w$, which corresponds to the Beltrami differential,

$$\frac{w}{\operatorname{Im} z} \nu_z,$$

which in turn corresponds to the tangent vector,

$$-2iw \frac{\partial}{\partial z}.$$

Thus, identifying $(QP_D)^*$ with TP_D, we have

$$\frac{i}{2}\Phi(z,w) = \left(z, w\frac{\partial}{\partial z}\right). \tag{9.3}$$

We have,

$$\begin{pmatrix} 1 & \operatorname{Re} z \\ 0 & \operatorname{Im} z \end{pmatrix} \cdot \operatorname{Sum}((X,\omega), I_{(u+iv)^2}) = \operatorname{Sum}((X_z,\omega_z), I_{(u+zv)^2}),$$

so in (z,w) coordinates, the leaf of $\mathcal{F}_D$ through $(i,(u+iv)^2)$ is parameterized by $z \mapsto (z,(u+zv)^2)$, which corresponds to the leaf of $\mathcal{C}_{P_D}$ through $\frac{i}{2}\Phi(i,w)$.

The proof in the case where $C = W_D$ is nearly identical except that we use the operation of splitting a double zero to parameterize a tubular neighborhood of W_D. Recall that a splitting of a double zero of a differential (X,ω) is determined by an embedded "X" as defined in §3. Equation (9.2) becomes

$$f(z,w) = \operatorname{Split}((X_z,\omega_z), E_z(w)),$$

where E_z is a holomorphic map from some small disk $\Delta_\epsilon \subset \mathbb{C}$ to the space of embedded "X"s on (X_z,ω_z) such that for each segment I of $E_z(w)$,

$$\left(\int_I \omega_z\right)^2 = w^3.$$

Then, (9.3) becomes

$$\frac{i}{2}\Phi(z,w) = \left(z, w^3\frac{\partial}{\partial z}\right).$$

■

Theorem 9.5. *We have,*

$$[P_D]\cdot[\overline{\mathcal{F}}_D] = [W_D]\cdot[\overline{\mathcal{F}}_D] = 0. \tag{9.4}$$

Proof. Let Ψ be the form defined in §2 which represents the Thom class on $T\overline{P}_D(-D)$ or $T\overline{W}_D(-D)$, where D is the divisor of cusps. By Proposition 9.3, $\Phi^*\Psi$ represents either the cohomology class $[\overline{P}_D]$ or $3[\overline{W}_D]$. Then we obtain (9.4) from Proposition 9.4 and Theorem 2.3. ■

10 Intersection with boundary curves

In this section, we will calculate the intersection numbers of the curves C_P with the class $[\overline{\mathcal{F}}_D]$. We will show:

Theorem 10.1. *For any prototype $P = (a,b,c,\overline{q}) \in \mathcal{Y}_D$,*

$$[C_P]\cdot[\overline{\mathcal{F}}_D] = \begin{cases} \dfrac{\gcd(a,c)}{2\gcd(a,b,c)}\left(1 - \dfrac{b}{\sqrt{D}}\right), & \textit{if } a-b+c<0; \\ \dfrac{\gcd(a,c)}{2\gcd(a,b,c)}\left(1 - \dfrac{a+c}{\sqrt{D}}\right), & \textit{if } a-b+c>0. \end{cases}$$

From this, we will derive Theorem 1.2.

Definitions. We regard $U_P \subset Y_D^c$ as having coordinates (y_1, y_2, y_3) defined by (6.7) and Theorem 6.4. Recall we defined a cover $\widehat{U}_P$ of U_P having coordinates (y_2, y_3) given by (6.8). Let $\widetilde{U}_P \cong \mathbb{H} \times \mathbb{H}$ be the universal cover of U_P with natural coordinates (y_2, y_3). In each case, y_i ranges in either $\mathbb{H}$ or $\mathbb{H}/\mu\mathbb{Z}$ for some $\mu > 0$. We set

$$y_j = u_j + iv_j.$$

We now define several objects on U_P and its covers. We use the convention that for any object X on U_P, such as a subset, a foliation, or a form, $\widehat{X}$ denotes the pullback of X to $\widehat{U}_P$, and $\widetilde{X}$ denotes the pullback to $\widetilde{U}_P$.

- Define
  $$W = \{(y_1, y_2, y_3) \in U_P : \operatorname{Im} y_2 = 1\}.$$
  The locus W limits on a closed leaf of q_{P+} in C_{P+} and should be regarded as a boundary of a tubular neighborhood of C_P in U_P. We have
  $$\widehat{W} \cong \mathbb{R}/\frac{a'}{\gcd(a', c')}\lambda\mathbb{Z} \times \mathbb{R}/\frac{a'}{\gcd(a', a' + b' + c')}(\lambda - 1)\mathbb{Z} \times \mathbb{R}^+$$
  with coordinates (u_2, u_3, v_3).
- Define
  $$W^0 = \{(y_1, y_2, y_3) \in U_P : y_2 = i\} \subset W$$
  We have
  $$\widehat{W}^0 \cong \mathbb{R}/\frac{a'}{\gcd(a', a' + b' + c')}(\lambda - 1)\mathbb{Z} \times \mathbb{R}^+$$
  with coordinates (u_3, v_3).
- Define
  $$W_x = \{(y_1, y_2, y_3) \in U_P : \operatorname{Im} y_2 = 1 \text{ and } \operatorname{Im} y_3 = x\} \subset W.$$
  We have
  $$\widehat{W}_x \cong \mathbb{R}/\frac{a'}{\gcd(a', c')}\lambda\mathbb{Z} \times \mathbb{R}/\frac{a'}{\gcd(a', a' + b' + c')}(\lambda - 1)\mathbb{Z}$$
  with coordinates (u_2, u_3).
- The lift $\widetilde{\mathcal{A}}_D$ of the foliation $\mathcal{A}_D$ to $\widetilde{U}_P$ is given by
  $$y_2 + y_3 = \text{const}.$$
  We define the leaf,
  $$\widetilde{\mathcal{A}}_D(r) = \{(y_2, y_3) \in \mathbb{H} \times \mathbb{H} : y_2 + y_3 = r\}.$$
- Let ν_P be a 2-form on Y_D^c which is Poincaré dual to C_P, and whose support is compact and disjoint from C_P.

- Let $p\colon U_P \to W$ be the canonical projection, defined by

$$\pi(X,\omega) = a_t \cdot (X,\omega),$$

where t is the unique real number so that $a_t \cdot (X,\omega) \in W$.

- Let $\eta_P = p_* \nu_P$, a 1-form on W.
- Let $A(r)$ be the area of any surface $(X,\omega) \in \widetilde{\mathcal{A}}_D(r)$.

Induced foliation of W. We now study the intersection of W with $\mathcal{F}_D$ and the induced measured foliation of W.

Proposition 10.2. *W is transverse to $\mathcal{F}_D$.*

Proof. At $p \in W$, the tangent space $T_p Y_D$ is spanned by $T_p W$ and the tangent to the curve $a_t \cdot p$. ■

Let $\mathcal{W}$ be the induced measured foliation of W.

Proposition 10.3. *Each leaf of $\mathcal{W}$ is contained in some W_x. The induced foliation $\mathcal{W}_x$ of W_x is the foliation by lines of slope x. More precisely, $\mathcal{W}_x$ is generated by the vector field*

$$\frac{\partial}{\partial u_2} + x \frac{\partial}{\partial u_3}.$$

The cylinder $W^0 \subset W$ is transverse to $\mathcal{W}_x$.

Proof. The leaf of $\widetilde{W}$ through $p = (u_2 + i, u_3 + iv_3) \in \widetilde{W}$ is $\{h_t \cdot p\}$. So the tangent vector to $\mathcal{W}$ at p,

$$\frac{d}{dt} h_t \cdot p = \frac{\partial}{\partial u_2} + v_3 \frac{\partial}{\partial v_3},$$

is the required tangent to $\widetilde{W}_{v_3}$. The transversality statement is then clear. ■

Recall that we have local coordinates (u_3, v_3) on W^0.

Proposition 10.4. *The measure on W^0 induced by the transverse measure to $\mathcal{W}$ is*

$$\frac{a}{\sqrt{D}} \frac{1}{(v_3+1)^3} du_3 dv_3 \tag{10.1}$$

Proof. Recall that we defined in §4 a leafwise measure on $\mathcal{A}_D$ which is invariant under the holonomy of $\mathcal{F}_D$. The leaf $\widetilde{\mathcal{A}}_D(i)$ is parameterized by the coordinates (u_3, v_3), and in these coordinates, the measure on $\widetilde{\mathcal{A}}_D(i)$ is

$$\mu = \frac{1}{\sqrt{A(1)}} du_3 dv_3$$

(the constant factor is to normalize the surfaces parameterized by this leaf to have unit area).

Let $\mathrm{Hol}\colon \widetilde{W}^0 \to \widetilde{\mathcal{A}}_D(i)$ be the holonomy map for $\widetilde{F}_D$. We have

$$\begin{pmatrix} 1 & -\dfrac{u_3}{v_3+1} \\ 0 & \dfrac{1}{v_3+1} \end{pmatrix} \cdot (i, u_3 + iv_3) = \left(\frac{-u_3 + i}{v_3+1}, \frac{u_3 + iv_3}{v_3+1} \right) \in \widetilde{\mathcal{A}}_D(i),$$

so

$$\mathrm{Hol}(u_3 + iv_3) = \frac{u_3 + iv_3}{v_3 + 1}$$

in the coordinates (u_3, v_3) on the domain and range. Thus the induced measure on W^0 is

$$\frac{1}{A(1)} |D \,\mathrm{Hol}\,| du_3 dv_3 = \frac{1}{A(1)(v_3+1)^3} du_3 dv_3,$$

which is equal to (10.1) by the following Lemma. ∎

Lemma 10.5. *We have*

$$A(r) = \frac{\sqrt{D}}{a} \operatorname{Im} r. \tag{10.2}$$

Proof. Consider $(X_p, \omega_p) \in \widetilde{\mathcal{A}}_D(r)$ with $v_2 + v_3 = \operatorname{Im} r$. We have

$$\mathrm{Area}(X_p, \omega_p) = v_1 + \lambda v_2 + (\lambda - 1) v_3. \tag{10.3}$$

Using (6.8) and the equations,

$$\lambda - \frac{c}{a\lambda} = \lambda - 1 - \frac{a+b+c}{a(\lambda-1)} = \frac{\sqrt{D}}{a},$$

which follow from elementary algebra we see that (10.3) reduces to (10.2). ∎

Integral of ν_P over U_P. For any $t \in \mathbb{Q} \cdot (\lambda - 1)/\lambda$, the foliation $\widehat{\mathcal{W}}_t$ of $\widehat{W}_t$ consists of closed leaves. Let δ_t be one of these closed leaves, and let Δ_t be the leaf of $\widehat{\mathcal{F}}_D$ containing δ_t, a punctured disk.

Proposition 10.6. *Given*

$$t = \frac{p}{q} \frac{\gcd(a', c')}{\gcd(a', a' + b' + c')} \frac{\lambda - 1}{\lambda} \tag{10.4}$$

with $\gcd(p, q) = 1$*, we have*

$$\int_{\delta_t} \widehat{\eta}_P = q, \quad \textit{and} \tag{10.5}$$

$$\int_{\delta_t} \widehat{\eta}_{P^+} = p. \tag{10.6}$$

Proof. Define

$$u = e^{2\pi i(\gcd(a',c')/a'\lambda)y_2} \quad \text{and}$$
$$v = e^{2\pi i(\gcd(a',a'+b'+c')/a'(\lambda-1))y_3}.$$

These (u, v) are the same coordinates as in Theorem 7.7 on the polydisk Δ^2 which parameterizes a neighborhood of c_P in Y_D^c by $f\colon \Delta^2 \to Y_D^c$. Define

$$\widehat{C}_P = f^{-1}(C_P) = \{(u,v) \in \Delta^2 \colon u = 0\}$$
$$\widehat{C}_{P+} = f^{-1}(C_{P+}) = \{(u,v) \in \Delta^2 \colon v = 0\}.$$

We have

$$\int_{\delta_t} \widehat{\eta}_P = \int_{\Delta_t} \widehat{\nu}_P = \overline{\Delta}_t \cdot \widehat{C}_P \quad \text{and}$$
$$\int_{\delta_t} \widehat{\eta}_{P+} = \int_{\Delta_t} \widehat{\nu}_{P+} = \overline{\Delta}_t \cdot \widehat{C}_{P+},$$

where the last expressions are the local intersection numbers of these curves at $(0,0) \in \Delta^2$. In (u, v)-coordinates,

$$\widehat{W}_t = \{(u,v) \in \Delta^2 : |u| = r \text{ and } |v| = s\},$$

for some $0 < r, s < 1$. Let $S \subset \Delta^2$ be the sphere of radius $\sqrt{r^2 + s^2}$, which contains $\widehat{W}_t$. We have

$$\overline{\Delta}_t \cdot \widehat{C}_P = \operatorname{Link}(\delta_t, \zeta)$$

where Link denotes the linking number, and $\zeta = S \cap \widehat{C}_P$. In S, the curve ζ bounds a disk which intersects $\widehat{W}_t$ in the curve $\xi = \{u_2 = 0\}$. Thus

$$\operatorname{Link}(\delta_t, \zeta) = \delta_t \cdot \xi = q$$

(where the last expression is the intersection number in $\widehat{W}_t$) because δ_t is a geodesic on $\widehat{W}_t$ with its flat metric of slope p/q. Thus we obtain (10.5) and (10.6) follows in the same way. ■

Proposition 10.7. *We have*

$$\int_{U_P} \nu_P = \frac{1}{2}\frac{a}{\sqrt{D}}\frac{\gcd(a,c)}{\gcd(a,b,c)}(\lambda - 1), \quad \textit{and} \tag{10.7}$$
$$\int_{U_P} \nu_{P+} = \frac{1}{2}\frac{a}{\sqrt{D}}\frac{\gcd(a,a+b+c)}{\gcd(a,b,c)}\lambda. \tag{10.8}$$

Proof. Give $\widehat{\mathcal{W}}_t$ the transverse measure du_3. With t as in (10.4), a segment transverse to $\widehat{\mathcal{W}}_t$ of transverse measure

$$\frac{1}{q}\frac{a'}{\gcd(a', a'+b'+c')}(\lambda - 1)$$

meets each leaf of $\widehat{\mathcal{W}}_t$ once, so by Proposition 10.6,

$$\begin{aligned}
\int_{\widehat{\mathcal{W}}_t} \widehat{\eta}_P &= \frac{a'}{\gcd(a', a'+b'+c')}(\lambda - 1), \quad \text{and} \\
\int_{\widehat{\mathcal{W}}_t} \widehat{\eta}_{P^+} &= \frac{p}{q}\frac{a'}{\gcd(a', a'+b'+c')}(\lambda - 1) \\
&= t\frac{a'}{\gcd(a', c')}\lambda.
\end{aligned}$$

By continuity, these formulas hold for all $t \in \mathbb{R}^+$. By the definition of η_P, we have

$$\int_{\mathcal{F}_D|U_P} \nu_P = \int_{\mathcal{W}} \eta_P,$$

and similarly for ν_{P^+}. Finally, from Fubini's Theorem, Proposition 10.4 and the fact that $\widehat{U}_P$ is an $a'/\gcd(a',c')\gcd(a', a'+b'+c')$-fold cover of U_P, we obtain

$$\begin{aligned}
\int_{\mathcal{F}_D|U_P} \nu_P &= \frac{\gcd(a',c')\gcd(a', a'+b'+c')}{a'} \int_{\widehat{\mathcal{W}}} \widehat{\eta}_P \\
&= \frac{a}{\sqrt{D}}\gcd(a',c')(\lambda - 1)\int_0^\infty \frac{dv_3}{(1+v_3)^3} \\
&= \frac{a}{2\sqrt{D}}\gcd(a',c')(\lambda - 1),
\end{aligned}$$

and similarly

$$\int_{U_P} \nu_{P^+} = \frac{a}{2\sqrt{D}}\gcd(a', a'+b'+c')\lambda.$$

■

Proof of Theorem 10.1. Since ν_P is supported on the locus of periodic surfaces, and the locus of two-cylinder surfaces has measure zero, we have

$$[C_P]\cdot[\overline{\mathcal{F}}_D] = \int_{\mathcal{F}_D|U_P} \nu_P + \int_{\mathcal{F}_D|U_{P^-}} \nu_P. \tag{10.9}$$

There are two cases to consider. When $a - b + c < 0$, then by the definition of P^- and by Proposition 10.7, (10.9) becomes

$$[C_P]\cdot[\overline{\mathcal{F}}_D] = \frac{1}{2}\left(\frac{a}{\sqrt{D}}(\lambda - 1) + \frac{a}{\sqrt{D}}(\lambda + 1)\right)\frac{\gcd(a,c)}{\gcd(a,b,c)}.$$

When $a - b + c > 0$, (10.9) becomes

$$[C_P]\cdot[\overline{\mathcal{F}}_D] = \frac{1}{2}\left(\frac{a}{\sqrt{D}}(\lambda - 1) - \frac{c}{\sqrt{D}}\frac{\lambda + 1}{\lambda}\right)\frac{\gcd(a,c)}{\gcd(a,b,c)}.$$

Both formulas reduce to the desired expression by elementary algebra. ■

Corollary 10.8. *For every prototype $P \in \mathcal{Y}_D$,*

$$[C_P] \cdot ([\overline{\mathcal{F}}_D] + \tau^*[\overline{\mathcal{F}}_D]) = [C_P] \cdot [\overline{P}_D].$$

Proof. From Theorem 10.1, and from the definition of the involution t on $\mathcal{Y}_D$, we have

$$[C_P] \cdot [\overline{\mathcal{F}}_D] + [C_{t(P)}] \cdot [\overline{\mathcal{F}}_D] = \frac{\gcd(a,c)}{\gcd(a,b,c)}.$$

By Theorem 7.5, this is just $[C_P] \cdot [\overline{P}_D]$. ∎

Volume of $E_D(1,1)$. We are now ready to calculate the volume of $E_D(1,1)$ with respect to μ_D. In [Bai], we showed:

Theorem 10.9. *The fundamental classes of $\overline{W}_D$ and $\overline{P}_D$ in $H^2(Y_D;\mathbb{Q})$ are given by*

$$[\overline{W}_D] = \frac{3}{2}[\omega_1] + \frac{9}{2}[\omega_2] + B_D, \quad \textit{and}$$
$$[\overline{P}_D] = \frac{5}{2}[\omega_1] + \frac{5}{2}[\omega_2] + B_D,$$

where $B_D \in H^2(Y_D;\mathbb{Q})$ is a linear combination of the fundamental classes of the curves C_P.

We also showed in [Bai, Lemma 13.1]:

Lemma 10.10. *The self-intersection number of B_D is*

$$B_D \cdot B_D = -15\chi(X_D).$$

Proof. This can be seen easily from Theorem 10.9 using the fact that

$$[\overline{W}_D] \cdot [\overline{P}_D] = 0$$

because these curves are disjoint by Theorem 7.5. ∎

Theorem 10.11. *The volume of $E_D(1,1)$ with respect to μ_D is*

$$\mathrm{vol}(E_D(1,1)) = 4\pi\chi(X_D).$$

Proof. By Theorem 10.9,

$$[\omega_1] = -\frac{1}{3}[\overline{W}_D] + \frac{3}{5}[\overline{P}_D] - \frac{4}{15}B_D.$$

Then by Theorem 9.5,

$$[\omega_1] \cdot [\overline{\mathcal{F}}_D] = -\frac{4}{15}B_D \cdot [\overline{\mathcal{F}}_D].$$

By Proposition 4.3 and Theorem 10.9, $\tau^* B_D = B_D$. We then have

$$\begin{aligned}
B_D \cdot [\overline{\mathcal{F}}_D] &= \frac{1}{2}(B_D + \tau^* B_D) \cdot [\overline{\mathcal{F}}_D] \\
&= \frac{1}{2} B_D \cdot ([\overline{\mathcal{F}}_D] + \tau^*[\overline{\mathcal{F}}_D]) \\
&= \frac{1}{2} B_D \cdot [P_D] && \text{(by Corollary 10.8)} \\
&= \frac{1}{2} B_D \cdot B_D && \text{(by Theorem 10.9)} \\
&= -\frac{15}{2}\chi(X_D). && \text{(by Lemma 10.10)}
\end{aligned}$$

Thus $[\omega_1] \cdot [\overline{\mathcal{F}}_D] = 2\chi(X_D)$, and the claim follows by Theorem 8.7. ■

11 Counting functions

In this section, we calculate the integrals over $\Omega_1 E_D(1,1)$ of the counting functions $N_{\mathrm{c}}(T, L)$ and $N_{\mathrm{s}}^i(T, L)$ defined in §1, proving Theorem 1.3.

Integral of $N_{\mathrm{c}}(T, L)$. Given a prototype $P = (a, b, c, \overline{q})$, define

$$v(P) = a'\lambda(\lambda - 1)\left(1 + \frac{1}{\lambda^2} + \frac{1}{(\lambda-1)^2}\right).$$

Theorem 11.1. *For any $\epsilon > 0$,*

$$\int_{\Omega_1 E_D(1,1)} N_c(T, \epsilon)\, d\mu_D^1(T) = \epsilon^2 \sum_{P \in \mathcal{Y}_D} v(P).$$

Proof. Since $N_c(T, \epsilon)$ is rotation-invariant, we can regard $N_c(T, \epsilon)$ as a function on X_D or X_D^c, where as a function on X_D^c, $N_c((X, \omega), \epsilon)$ is the number of cylinders in the horizontal foliation of (X, ω) which have circumference at most ϵ, with (X, ω) normalized to have unit area. We then have

$$\begin{aligned}
\int_{\Omega_1 E_D(1,1)} N_c(T, \epsilon)\, d\mu_D^1(T) &= \sum_{c \in C(X_D)} \int_{X_D^c} N_c(T, \epsilon)\, d\mu_D(T) \\
&= \sum_{P \in \mathcal{Y}_D} \int_{U_P} N_c(T, \epsilon)\, d\mu_D(T).
\end{aligned}$$

We will show that

$$\int_{U_P} N_c(T, \epsilon)\, d\mu_D(T) = \epsilon^2 v(P). \tag{11.1}$$

We continue to use the notation of §10, in particular the locus $W \subset U_P$ and the coordinates $y_j = u_j + i v_j$ on U_P. Given $p = (X, \omega) \in W$ and $L \in \mathbb{R}^+$, define

$$B_L = \{a_s h_t p : s \in \mathbb{R} \text{ and } 0 \leq t \leq L\},$$

a subset of a leaf of $\mathcal{F}_D$ which we regard as a copy of $\mathbb{H}$ with its hyperbolic area measure ρ. We identify the point $a_t h_s p$ of this leaf with $e^t i + s \in \mathbb{H}$.

We claim that

$$\int_{B_L} N_c(T, \epsilon)\, d\rho(T) = \epsilon^2 \frac{\sqrt{D}}{a}(1+v_3)\left(1 + \frac{1}{\lambda^2} + \frac{1}{(\lambda-1)^2}\right) L. \tag{11.2}$$

The form (X, ω), normalized to have unit area has three horizontal cylinders of circumference $1/\sqrt{A}$, $\lambda/\sqrt{A}$, and $(\lambda-1)/\sqrt{A}$, where

$$A = \frac{\sqrt{D}}{a}(1+v_3)$$

by Lemma 10.5. A cylinder of circumference ℓ on (X, ω) has circumference less than ϵ on $a_t h_s(X, \omega)$ if and only if

$$e^t > \frac{\ell^2}{\epsilon^2 A},$$

so the contribution of this cylinder to (11.2) is

$$\int_0^L \int_{\ell^2/\epsilon^2 A}^{\infty} \frac{1}{y^2}\, dy\, dx = \frac{\epsilon^2 A L}{\ell^2},$$

which yields (11.2) by summing over all cylinders.

Now, with $p\colon U_P \to W$ the canonical projection along a_s-orbits,

$$\sigma = p_*(N_c(\cdot, \epsilon)\rho)$$

is a leafwise measure for the measured foliation $\mathcal{W}$. If we parameterize the leaves of $\mathcal{W}$ as unit-speed horocycles, we obtain the leafwise measure du_2, so by (11.2),

$$\sigma = \epsilon^2 \frac{\sqrt{D}}{a}(1+v_3)\left(1 + \frac{1}{\lambda^2} + \frac{1}{(\lambda-1)^2}\right) du_2.$$

Let τ be the transverse measure to $\mathcal{W}$, given by Proposition 10.4. We have

$$\begin{aligned}
\int_{U_P} N_c(T, \epsilon)\, d\mu_D(T) &= \int_W d\sigma\, d\tau \\
&= \int_0^\infty \int_{W_{v_3}} \epsilon^2 \left(1 + \frac{1}{\lambda^2} + \frac{1}{(\lambda-1)^2}\right) \frac{1}{(1+v_3)^2}\, du_2\, du_3\, dv_3 \\
&= \epsilon^2 a' \lambda(\lambda-1)\left(1 + \frac{1}{\lambda^2} + \frac{1}{(\lambda-1)^2}\right) \int_0^\infty \frac{1}{(1+v_3)^2} dv_3 \\
&= \epsilon^2 v(P),
\end{aligned}$$

which proves (11.1). ■

Sums over prototypes. We now evaluate the sum over prototypes appearing in Theorem 11.1. Define an involution $s\colon \mathcal{Y}_D \to \mathcal{Y}_D$ by

$$s(P) = t(P^+),$$

or equivalently

$$s(a, b, c, \overline{q}) = (a, -2a - b, a + b + c, \overline{q}).$$

We also make the following definitions:

$$\begin{aligned}
&v'(P) = a\lambda(\lambda - 1)\left(1 + \frac{1}{\lambda^2} + \frac{1}{(\lambda-1)^2}\right),\\
&w(P) = v'(P) + v'(s(P)),\\
&S_D = \{(a, b, c) \in \mathbb{Z}^3 : b^2 - 4ac = D, a > 0, c < 0, \text{ and } a + b + c < 0\}, \quad \text{and}\\
&S'_D = \{(a, b, c) \in \mathbb{Z}^3 : b^2 - 4ac = D, a > 0, \text{ and } c < 0\}.
\end{aligned}$$

Since the definitions of v, v', w, and s make sense on elements of S_D, we regard them to be defined there as well.

Theorem 11.2. *For any quadratic discriminant D,*

$$\sum_{P \in \mathcal{Y}_D} v(P) = 60\chi(X_D). \tag{11.3}$$

Lemma 11.3. *Given a quadratic discriminant $D = f^2E$ with E fundamental and $f \in \mathbb{N}$, we have*

$$\sum_{s|f} \sum_{P \in \mathcal{Y}_{s^2E}} v(P) = \sum_{P \in S_D} v'(P). \tag{11.4}$$

Proof. Given $P = (a, b, c) \in S_D$, note that for any discriminant D/s^2 prototype $Q = (a/s, b/s, c/s, \overline{q})$, we have $v(P) = v(Q)$. We need to show that the contribution to the left hand side of (11.4) from prototypes Q of this form is $v'(P)$. If $s|\gcd(a, b, c)$, then there are $\phi(\gcd(a, b, c)/s)$ prototypes in $\mathcal{Y}_{D/s^2}$ of the form $(a/s, b/s, c/s, \overline{q})$ (where ϕ is the Euler function). The contribution to the left hand side from $P = (a, b, c)$ is

$$v(P) \sum_{s|\gcd(a,b,c)} \phi\left(\frac{\gcd(a, b, c)}{s}\right) = v(P)\gcd(a, b, c) = v'(P).$$

■

The following lemma is just a computation in elementary algebra and will be left to the reader.

Lemma 11.4. *For any $P = (a, b, c) \in S_D$,*

$$w(P) = 4a + b + \frac{b^2}{a} + \frac{ab}{c} - 2c - \frac{a(2a + b)}{a + b + c}.$$

Lemma 11.5. *The following identities hold:*

$$\sum (a+b) = 0 \tag{11.5}$$

$$\sum \frac{ab}{c} = -\sum \frac{a(2a+b)}{a+b+c} \tag{11.6}$$

$$\sum \frac{2bc}{a} + \frac{b^2}{a} - a + 2c = 0 \tag{11.7}$$

$$\sum \frac{ab}{c} = \sum \frac{bc}{a} \tag{11.8}$$

$$\sum (a-c) = -5H(2,D). \tag{11.9}$$

All sums are over $(a,b,c) \in S_D$.

Proof. Applying the involution s, we have

$$\sum a+b = \sum a + (-2a-b) = -\sum a+b,$$

which proves (11.5).

By applying s to $\sum ab/c$, we obtain (11.6) immediately.

Applying the involution s and (11.5) yields

$$\begin{aligned} \sum \frac{bc}{a} &= -\sum 2a+3b+2c+\frac{b^2}{a}+\frac{bc}{a} \\ &= \sum a-2c-\frac{b^2}{a}-\frac{bc}{a}, \end{aligned}$$

which is (11.7).

The sum,

$$\sum \frac{ab}{c} - \frac{bc}{a} = b\sum \frac{a}{c} - \frac{c}{a},$$

is invariant under the involution,

$$\sigma(a,b,c) = (-c,-b,-a),$$

of S'_D. Since S'_D is the disjoint union $S_D \cup \sigma(S_D)$, this implies

$$\sum_{(a,b,c)\in S_D} \frac{ab}{c} - \frac{bc}{a} = \frac{1}{2} \sum_{(a,b,c)\in S'_D} b\left(\frac{a}{c} - \frac{c}{a}\right) = 0,$$

where the second sum vanishes because the elements (a,b,c) and $(a,-b,c)$ contribute opposite values to this sum. This implies (11.8).

Since the summand in (11.9) is also σ-invariant, we have

$$\begin{aligned}
\sum_{P\in S_D} a-c &= \frac{1}{2}\sum_{P\in S'_D} a-c && \text{(because the sum is } \sigma\text{-invariant)}\\
&= \sum_{P\in S'_D} a && \text{(applying } \sigma \text{ again)}\\
&= \sum_{e\equiv D\ (2)} \sigma_1\left(\frac{D-e^2}{4}\right)\\
&= -5H(2,D), && \text{(by Theorem 4.8)}
\end{aligned}$$

which proves (11.9). ■

Proof of Theorem 11.2. We have

$$\begin{aligned}
\sum w(P) &= \sum 3a + \frac{b^2}{a} + \frac{2ab}{c} - 2c && \text{(Lemma 11.4, (11.5), and (11.6))}\\
&= \sum 4a - 4c - \frac{2bc}{a} + \frac{2ab}{c} && \text{(by (11.7))}\\
&= \sum 4a-4c && \text{(by (11.8))}\\
&= -20H(2,D) && \text{(by (11.9)),}
\end{aligned}$$

with all sums over $(a,b,c)\in S_D$. Thus

$$\sum v'(P) = \frac{1}{2}\sum w(P) = -10H(2,D),$$

and we obtain (11.3) from Lemma 11.3, Theorem 4.7, (4.3), and Möbius inversion. ■

This completes the proof of (1.3).

Integral of $N_{\mathrm{s}}^i(T,L)$. We now compute the integrals of the counting functions $N_{\mathrm{s}}^i(T,L)$ over $\Omega_1 E_D(1,1)$, using the operation of collapsing a saddle connection to relate this to the volumes of $\Omega_1 P_D$ and $\Omega_1 W_D$.

Define S_i to be the space of pairs (T,I), where $T\in\Omega_1 E_D(1,1)$ and I is a multiplicity i saddle connection on T joining distinct zeros. Let $p_i\colon S_i\to\Omega_1 E_D(1,1)$ be the natural local homeomorphism forgetting the saddle connection. We equip S_i with the measure $\sigma_i = p^*\mu_D^1$ which is determined by $\sigma_i(U)=\mu_D^1(U)$ if p_i is injective on U. Given $\epsilon>0$, let $S_i(\epsilon)$ be the locus of saddle connections of length less than ϵ.

Lemma 11.6. *For every $\epsilon>0$,*

$$\mathrm{vol}_{\sigma_i} S_i(\epsilon) = \int\limits_{\Omega_1 E_D(1,1)} N_{\mathrm{s}}^i(T,\epsilon)\,d\mu_D^1(T). \tag{11.10}$$

Proof. Every point T in $\Omega_1 E_D(1,1)$ has exactly $N_{\mathrm{s}}^i(T,\epsilon)$ preimages under p_i. ■

Recall that the operation of collapsing a saddle connection is defined if it is unobstructed in the sense of §3.

Lemma 11.7. *Almost every saddle connection in S_2 is unobstructed.*

Proof. It suffices to show that almost every surface in $\Omega_1 E_D(1,1)$ has no obstructed saddle connection, which in turn would follow from the same statement for $\Omega E_D(1,1)$. If a saddle connection I on (X,ω) is obstructed, then from the definition of an obstruction, there must necessarily be an absolute period of ω which is a real multiple of the relative period $\omega(I)$.

Let $R \subset \Omega_1 E_D(1,1)$ be the locus of surfaces which have a relative period and an absolute period which are real multiples of each other. It suffices to show that R has measure zero. Consider $U \subset \Omega\mathcal{M}_2$ with period coordinates $U \to \mathbb{C}^5$ defined in §3. In these coordinates, $\Omega E_D(1,1)$ is a codimension two linear subspace. The locus R is a countable union of real-linear subspaces. Since we can vary any relative period while keeping the absolute periods constant by moving along the kernel foliation, each of these subspaces is proper, thus R has measure zero. ■

Lemma 11.8. *For any $\epsilon > 0$, we have*

$$\operatorname{vol}_{\sigma_1} S_1(\epsilon) = \frac{\pi\epsilon^2}{2} \operatorname{vol} \Omega_1 P_D, \quad \textit{and}$$
$$\operatorname{vol}_{\sigma_2} S_2(\epsilon) = \frac{3\pi\epsilon^2}{2} \operatorname{vol} \Omega_1 W_D.$$

Proof. Let $q_1 \colon S_1(\epsilon) \to \Omega_1 P_D \times (\Delta_\epsilon/\pm 1)$ and $q_2 \colon S_2(\epsilon) \to \Omega_1 W_D \times (\Delta_\epsilon/\pm 1)$ be defined by

$$q_i((X,\omega), I) = \left(\operatorname{Collapse}((X,\omega), I), \int_I \omega\right).$$

The map q_2 is really only defined almost everywhere, on the set of unobstructed saddle connections (using Lemma 11.7). The map q_1 is an isomorphism of measure spaces, since the connected sum operation provides a measurable inverse. The map q_2 is a local isomorphism of measure spaces and almost everywhere three-to-one, since the operation of splitting a zero provides a local inverse, and there are three ways to split a zero along almost every segment $\overline{0w} \subset \mathbb{C}$. Therefore,

$$\operatorname{vol}_{\sigma_1} S_1(\epsilon) = \operatorname{vol}(\Omega_1 P_D \times (\Delta_\epsilon/\pm 1)) = \frac{\pi\epsilon^2}{2} \operatorname{vol} \Omega_1 P_D, \quad \text{and}$$
$$\operatorname{vol}_{\sigma_2} S_2(\epsilon) = 3 \operatorname{vol}(\Omega_1 W_D \times (\Delta_\epsilon/\pm 1)) = \frac{3\pi\epsilon^2}{2} \operatorname{vol} \Omega_1 W_D.$$

■

Corollary 11.9. *For every $\epsilon > 0$, we have*

$$\int_{\Omega_1 E_D(1,1)} N_{\mathrm{s}}^1(T,\epsilon)\, d\mu_D^1(T) = \frac{27}{2}\pi^2\epsilon^2\chi(X_D), \quad \textit{and}$$
$$\int_{\Omega_1 E_D(1,1)} N_{\mathrm{s}}^2(T,\epsilon)\, d\mu_D^1(T) = \frac{5}{2}\pi^2\epsilon^2\chi(X_D).$$

Proof. This follows from Lemmas 11.6 and 11.8 together with

$$\begin{aligned}\operatorname{vol}\Omega_1 P_D &= -2\pi\chi(P_D) = 5\pi\chi(X_D)\\ \operatorname{vol}\Omega_1 W_D &= -2\pi\chi(W_D) = 9\pi\chi(X_D),\end{aligned}$$

which we proved in [Bai]. ∎

This completes the proof of Theorem 1.3.

12 Equidistribution of large circles

In this section, we prove Theorem 1.4, that circles in $\Omega_1 E_D(1,1)\backslash\Omega_1 D_{10}$ become equidistributed as their radius goes to infinity.

Let $\mathcal{E}_D$ be the two fold covering of $\Omega_1 E_D(1,1)$ whose points are eigenforms $(X,\omega) \in \Omega_1 E_D(1,1)$ together with a choice of ordering of the zeros of ω. In $\mathcal{E}_5$, let $\mathcal{D}$ be the inverse image of the decagon curve $\Omega_1 D_{10} \subset \Omega_1 E_D(1,1)$. The action of $\mathrm{SL}_2\mathbb{R}$ on $\Omega_1 E_D(1,1)$ lifts to an action of $\mathrm{SL}_2\mathbb{R}$ on $\mathcal{E}_D$, and there is a natural ergodic, finite, $\mathrm{SL}_2\mathbb{R}$-invariant measure ν_D on $\mathcal{E}_D$ obtained by pulling back the measure μ_D^1 on $\Omega_1 E_D(1,1)$.

Given $x \in \mathcal{E}_D$, let m_x be the uniform measure on $\mathrm{SO}_2\mathbb{R}\cdot x$ and let $m_x^t = (a_t)_* m_x$. Since ν_D projects to μ_D^1, Theorem 1.4 follows from the following theorem:

Theorem 12.1. *For any $x \in \mathcal{E}_D$ for $D \neq 5$ or $x \in \mathcal{E}_5 \setminus \mathcal{D}$,*

$$\lim_{t\to\infty} m_x^t = \frac{\nu_D}{\operatorname{vol}(\nu_D)}.$$

Lemma 12.2. *To prove Theorem 12.1, it suffices to show that for any subsequential limit $m_x^\infty = \lim m_x^{t_n}$, the measure m_x^∞ assigns zero mass to $\mathcal{D}$. In particular, Theorem 12.1 is true for any $x \in \mathcal{E}_D$ for $D \neq 5$.*

Proof. Let $\mathcal{E}_D^h \subset \mathcal{E}_D$ be the locus of surfaces which have a horizontal saddle connection. Points in $\mathcal{E}_D^h$ diverge under the action of a_t in the sense that for every compact set K of $\mathcal{E}_D^h$, the images $a_t(K)$ eventually leaves every compact set of $\mathcal{E}_D$.

Let M be the space of measures on $\mathcal{E}_D$ with the weak* topology, and let $S \subset M$ be the compact, a_t-invariant set of subsequential limits of m_x^t. By Corollary 5.3 of [EM01], every measure in S has unit mass.

It is well-known that any subsequential limit of m_x^t is h_t-invariant (see [EMM06, Lemma 7.3]). By the main result of [CW], every finite h_t-invariant measure on $\mathcal{E}_D$ is either supported on $\mathcal{E}_D^h$, is uniform measure on $\mathcal{D}$, or is the uniform measure ν_D on $\mathcal{E}_D$. But no measure $\mu \in S$ can be supported on $\mathcal{E}_D^h$ because by divergence we would have $(a_t)_*\mu \to 0$, but then $0 \in S$, which contradicts every measure in S having unit mass. ■

We now show that $\mathcal{D}$ is not a subsequential limit, following the proof of a similar statement in [EMM06].

Let $\mathcal{A}'_D$ be the kernel foliation of $\mathcal{E}_D$, that is, the foliation of $\mathcal{E}_D$ given by pulling back the foliation $\Omega_1\mathcal{A}_D$ of $\Omega_1 E_D(1,1)$. The leafwise quadratic differential q on $\Omega_1\mathcal{A}_D$ lifts to a leafwise Abelian differential on $\mathcal{A}'_D$, so leaves of $\mathcal{A}'_D$ have natural translation structures. Given $x \in \mathcal{E}_D$ and $v \in \mathbb{C}$, we write $x+v$ for the point in $\mathcal{E}_D$ obtained by translating x by v in its leaf of $\mathcal{A}'_D$. This is not always well defined because the segment through x in the direction of v might hit a cone point of the translation structure. The map $x \mapsto x+v$ is defined for almost every v and given x, for sufficiently small v. We have the fundamental identity,

$$A \cdot (x+v) = A\cdot x + Av$$

for every $A \in \mathrm{SL}_2\mathbb{R}$.

Now let $\mathcal{D}'$ be the Riemann surface orbifold of which $\mathcal{D}$ is the unit tangent bundle and let $\pi\colon \mathcal{D} \to \mathcal{D}'$ be the natural projection. Let $K \subset K' \subset \mathcal{D}$ be the inverse images under π of closed hyperbolic discs in $\mathcal{D}'$ with $K \subset \operatorname{int} K'$.

Given $r, s > 0$, define

$$\mathrm{Box}(r,s) = \{(x,y) \in \mathbb{R}^2 : |x| < r \text{ and } |y| < s\}.$$

Since K' is compact, the natural map $K' \times \mathrm{Box}(\rho,\rho) \to \mathcal{E}_D$ defined by $(x,v) \mapsto x+v$ is everywhere-defined and injective for some $\rho > 0$. Given any $S \subset K'$ and $r < \rho$, let $B_r(S)$ be the image of this map, which we identify with the box $S \times \mathrm{Box}(r,r)$. Define

$$R(S,r,t) = \{\theta \in \mathbb{R}/\mathbb{Z} : a_t r_\theta x \in B_r(S)\}.$$

Lemma 12.3. *For any $\delta > 0$, we have*

$$|R(K,\delta\rho,t)| < 2\delta|R(K',\rho,t)|$$

whenever t is sufficiently large.

Remark. We use the notation $|S|$ for the Lebesgue measure of S.

Proof. Let I be a component of $R(K',\rho,t)$. It suffice to show that

$$|I \cap R(K,\delta\rho,t)| < 2\delta|I|$$

for t sufficiently large. Suppose this intersection is nontrivial and that $\theta_0 \in I \cap R(K,\delta\rho,t)$. By the choice of ρ, there is a unique $v \in \mathrm{Box}(\delta\rho,\delta\rho)$ such that

$$y := a_t r_{\theta_0} x + v \in \mathcal{D}. \tag{12.1}$$

We define

$$\begin{aligned}
S &= \{\theta \in (\theta_0 - \pi/2, \theta_0 + \pi/2) : a_t r_\theta r_{\theta_0}^{-1} a_t^{-1} v \in \mathrm{Box}(\delta\rho, \delta\rho)\} \\
&= \{\theta \in (\theta_0 - \pi/2, \theta_0 + \pi/2) : r_{\theta-\theta_0} a_t^{-1} v \in \mathrm{Box}(e^{-t}\delta\rho, e^{t}\delta\rho)\}, \quad \text{and} \\
S' &= \{\theta \in (\theta_0 - \pi/2, \theta_0 + \pi/2) : a_t r_\theta r_{\theta_0}^{-1} a_t^{-1} v \in \mathrm{Box}(\rho, \rho)\}.
\end{aligned}$$

Let κ be the distance from x to $\mathcal{D}$ along $\mathcal{A}'_D$. We have $\kappa > 0$ since $x \notin \mathcal{D}$. Since $a_t^{-1} v + r_{\theta_0} x \in \mathcal{D}$, we have

$$|a_t^{-1} v| \geq \kappa. \tag{12.2}$$

Thus, if t is sufficiently large, we have

$$|a_t^{-1} v| > e^{-t}\rho. \tag{12.3}$$

It then follows from (12.3) and elementary trigonometry that

$$\begin{aligned}
\sin\frac{|S|}{2} &\leq \frac{e^{-t}\delta\rho}{|a_t^{-1} v|}, \quad \text{and} \\
\sin\frac{|S'|}{2} &= \frac{e^{-t}\rho}{|a_t^{-1} v|},
\end{aligned}$$

so

$$\begin{aligned}
|S| &\leq \frac{4e^{-t}\delta\rho}{|a_t^{-1} v|}, \quad \text{and} \\
\frac{2e^{-t}\rho}{|a_t^{-1} v|} \leq |S'| &\leq \frac{4e^{-t}\rho}{|a_t^{-1} v|}. \qquad (12.4)
\end{aligned}$$

Therefore,

$$\frac{|S|}{|S'|} \leq 2\delta. \tag{12.5}$$

We have $I \cap R(K, \delta\rho, t) \subset S$, so by (12.5), it is enough to show that $I = S'$. To show this, it is enough to show that for either endpoint ψ of S', we have

$$a_t r_\psi x \in \operatorname{int} K' \times \partial\, \mathrm{Box}(\rho, \rho), \tag{12.6}$$

identifying $B_\rho(K')$ with $K' \times \mathrm{Box}(\rho, \rho)$.

From (12.1), we have

$$a_t r_\psi x + a_t r_\psi r_{\theta_0}^{-1} a_t^{-1} v = a_t r_\psi r_{\theta_0}^{-1} a_t^{-1} y.$$

Let $y' = a_t r_\psi r_{\theta_0}^{-1} a_t^{-1} y$. The points $\pi(y)$ and $\pi(y')$ in $\mathcal{D}'$ are joined by an arc of angle $|\psi - \theta_0|$ in a hyperbolic circle of radius t. Therefore, the distance between $\pi(y)$ and $\pi(y')$ in $\mathcal{D}'$ is at most

$$|\psi - \theta_0| 2\pi \frac{e^t - 1}{e^{t/2}} < 8\pi e^{-t/2} \frac{\rho}{\kappa},$$

using (12.2) and (12.4). Since $\pi(K) \subset \operatorname{int} \pi(K')$, this is smaller than the distance between K and $\partial K'$ in $\mathcal{D}'$ if t is sufficiently large. This proves (12.6). ∎

Proof of Theorem 12.1. Let $m_x^\infty = \lim m_x^{t_n}$ for some sequence t_n. Suppose m_x^∞ assigns positive mass to $\mathcal{D}$. Then we can choose compact subsets $K \subset K' \subset \mathcal{D}$ as above so that $m_x^\infty(K) > 0$. For any $\delta > 0$, we have

$$\begin{aligned} m_x^\infty(K) &= \lim_{n\to\infty} m_x^{t_n}(K) \\ &\le \lim_{n\to\infty} m_t^{t_n} B_{\delta\rho}(K) \\ &= \lim_{n\to\infty} |R(K, \delta\rho, t_n)| \\ &\le \lim_{n\to\infty} 2\delta |R(K', \rho, t_n)| \\ &\le 2\delta, \end{aligned}$$

using Lemma 12.3. This contradicts $\mu_x^\infty(K) > 0$. Therefore, $m_x^\infty(\mathcal{D}) = 0$, which implies Theorem 12.1 by Lemma 12.2. ■

13 Applications to billiards

In this section, we summarize results of Eskin, Masur, and Veech which allow us to obtain the asymptotics for counting saddle connections and closed geodesics claimed in Theorems 1.1 and 1.5.

Counting problems. Consider the following situation. Let $S \subset \Omega_1\mathcal{M}_g$ be a stratum equipped with a finite, ergodic, $\mathrm{SL}_2\mathbb{R}$-invariant measure μ. Let $M(\mathbb{R}^2)$ be the space of measures on $\mathbb{R}^2$. Consider a function $V : S \to M(\mathbb{R}^2)$ which is $\mathrm{SL}_2\mathbb{R}$-equivariant and satisfies

$$N_V(T, R) < C N_{\mathrm{s}}(T, R)$$

for some constant C and any $T \subset S$, where $N_V(T, R)$ is the $V(T)$-measure of the ball $B_R(0) \subset \mathbb{R}^2$, and $N_{\mathrm{s}}(T, R)$ is the number of saddle connections of length at most R on T. We call the triple (S, μ, V) a *counting problem*.

Siegel-Veech transform. Given any measurable nonnegative $f : \mathbb{R}^2 \to \mathbb{R}$, the *Siegel-Veech transform* of f is the function $\widehat{f} : S \to \mathbb{R}$ defined by

$$\widehat{f}(T) = \int_{\mathbb{R}^2} f\, dV(T).$$

Note that if χ_R is the characteristic function of $B_R(0)$, then $\widehat{\chi}_R(T) = N_V(T, R)$.

Theorem 13.1 ([Vee98])**.** *There is a constant $c_{\mu,V}$ such that for any nonnegative, measurable f, we have*

$$\frac{1}{\mathrm{vol}(\mu)} \int_S \widehat{f}\, d\mu = c_{\mu,V} \int_{\mathbb{R}^2} f(x, y)\, dx\, dy.$$

The constant $c_{\mu,V}$ is called a *Siegel-Veech constant*.

Pointwise asymptotics. Given $T \in S$, let m_T be the uniform measure on $\mathrm{SO}_2\mathbb{R} \cdot T$. Eskin and Masur showed in [EM01]:

Theorem 13.2. *Consider a counting problem (S, μ, V). Suppose for some $T \in S$, we have*

$$\lim_{t\to\infty} (a_t)_* m_T = \frac{\mu}{\mathrm{vol}(\mu)}.$$

We then have

$$N_V(T, R) \sim \pi c_{\mu,V} R^2. \tag{13.1}$$

Remark. For T with nonunit area, the right hand side of (13.1) is scaled by a factor of $1/\operatorname{Area}(T)$.

Counting problems in genus two. Now restrict to the stratum $\Omega_1\mathcal{M}_2(1,1)$, equipped with the period measure μ_D^1 supported on $\Omega_1 E_D(1,1)$. Given a surface $T \in \Omega_1\mathcal{M}_2(1,1)$, let $\mathfrak{C}(T)$ be the set of maximal cylinders on T, and let $\mathfrak{S}_i(T)$ be the set of saddle connections joining distinct zeros of multiplicity i on T (we count pairs of saddle connections related by the hyperelliptic involution as a single saddle connection). A cylinder or saddle connection $I \in \mathfrak{C}(T)$ or $\mathfrak{S}_i(T)$ determines a holonomy vector $v(I) \in \mathbb{C}$, well-defined up to sign. Define

$$C(T) = \frac{1}{2} \sum_{I\in\mathfrak{C}(T)} (\delta_{v(I)} + \delta_{-v(I)}) \quad \text{and}$$

$$S_i(T) = \frac{1}{2} \sum_{I\in\mathfrak{S}_i(T)} (\delta_{v(I)} + \delta_{-v(I)}).$$

Proof of Theorem 1.5. Consider the counting problems $(\Omega_1\mathcal{M}_2(1,1), \mu_D^1, C)$ and $(\Omega_1\mathcal{M}_2(1,1), \mu_D^1, S_i)$ and let c_D and s_D^i be the associated Siegel-Veech constants. We calculate these constants by applying Theorem 13.1 with f the characteristic function χ_R. By Theorems 1.2 and 1.3, we have

$$c_D = \frac{15}{\pi^2},$$
$$s_D^1 = \frac{27}{8}, \quad \text{and}$$
$$s_D^2 = \frac{5}{8}.$$

We then obtain the desired asymptotics from Theorems 13.2 and 1.5. ∎

Unfolding. There is associated to every rational-angled polygon P a compact Riemann surface with a holomorphic one-form, called the *unfolding* $U(P)$ of P, obtained by gluing together several reflected copies of P.

If P is the L-shaped polygon with barrier $P(a, b, t)$ shown in Figure 1, then the unfolding $U(P)$ is the surface obtained by gluing four reflected copies of P as shown in Figure 6. It is easy to see that $U(P)$ is genus two with two simple zeros.

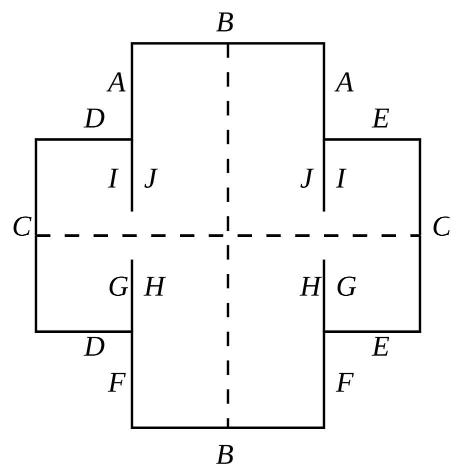

Figure 6: Unfolding of P

There is a natural refolding map $\pi\colon U(P) \to P$. The map π sends each closed geodesic of length L to a closed billiards path of length L. Conversely, each closed billiards path on P which is neither horizontal nor vertical is the image of exactly two closed geodesics on $U(P)$ of the same length. The analogous statement holds for saddle connections.

A saddle connection I on $U(P)$ has multiplicity one if and only if I passes through a Weierstrass point of $U(P)$. The six Weierstrass points of $U(P)$ map to the six corners of P. Therefore, a saddle connection on $U(P)$ is multiplicity one if and only if the corresponding saddle connection of P is multiplicity one. We therefore have

$$N_{\mathrm{c}}(U(P), R) \sim 2N_c(P, R), \tag{13.2}$$

and similarly for the counting functions for type one and two saddle connections.

It is straightforward to check that $U(P(x, y, t))$ lies on the decagon curve if and only if

$$x = y = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad t = \frac{5-\sqrt{5}}{10}.$$

We then obtain Theorem 1.1 directly from Theorem 1.5 and (13.2).

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